PROBLEM SECTION

You are invited to submit solutions to any or all of the following problems, accompanied by your name, school and year or form. Solutions of these problems will appear in the next issue of Parabola; your solution(s) may be used if they are received in time.

Q.817 Find all integers x, y such that $x(3y-5) = y^2+1$.

Q.818 If a, b, c are the side lengths of a triangle, A its area, and R the radius of the circumcircle, prove that $abc = 4AR$.

Q.819 The figure shows two ways of drawing six line segments joining the vertices of a convex hexagon in each a way that every pair of line segments intersect. It is impossible to draw seven line segments (each an unproduced side or diagonal of the hexagon) with this property. Prove that in fact it is not possible to draw $(n+1)$ sides and/or diagonals of a convex polygon with n vertices in such a way that every pair intersect.

Q.820 Show that, for any positive integer, n, the product of all integers between $n+1$ and $2n$ inclusive is equal to 2^n times the product of all the odd numbers less than $2n$.

Q.821 The convex quadrilateral $ABCD$ is not cyclic, and no two sides are parallel. How many circles can be drawn which are equidistant from all four vertices?

Q.822 If $0 < x_i < 1$ for $i = 1, 2, \cdots n$ and $x_1 x_2 \cdots x_n = (1-x_1)(1-x_2)\cdots(1-x_n)$ find (with proof) the maximum possible value of $P = x_1 x_2 \cdots x_n$.

Q.823 P is a point inside a triangle with sides of length a, b, c. The perpendicular distances from P to these sides are α, β, γ respectively. Prove that $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma}$ is not less than $\frac{(a+b+c)^2}{2\Delta}$ where Δ is the area of the triangle.

Q.824 $\{(m-n+1),\ (m-n+2), \cdots, (m-1), m\}$ is a set of n consecutive positive integers with the property that m is a factor of the least common multiple of $\{(m-n+1), \cdots, (m-1)\}$. Furthermore, there is no other set of n consecutive positive integers with the same property. Find m and n.

Q.825 Consider all subsets of 8 elements of the set $\{1, 2, 3, \cdots, 17\}$. From each subset select the smallest member. Show that the arithmetic mean of the 24,310 numbers selected is equal to 2.

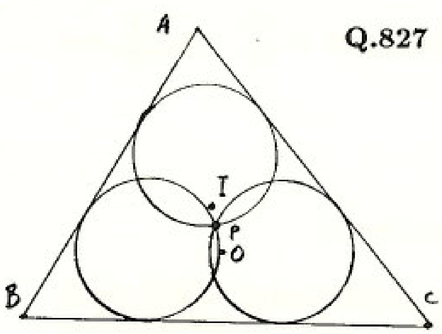
Q.826 The Fibonacci numbers are $\{1, 2, 3, 5, 8, 13, 21, \cdots\}$ where each number after the second is the sum of the previous two.

(i) Prove that every positive integer can be expressed as a sum of (one or more) distinct Fibonacci numbers.

(For example 21 can be so expressed in four different ways:- $21 = 21$; $21 = 13 + 8$; $21 = 13 + 5 + 3$; $21 = 13 + 5 + 2 + 1$).

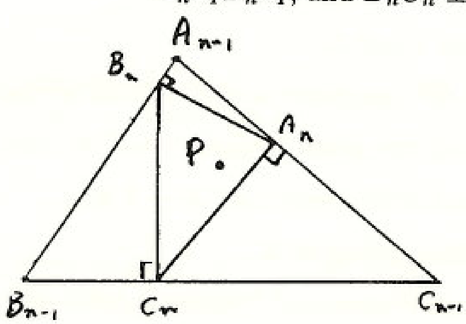
(ii) Find, with proof, all positive integers such that there is only one such expression.

(e.g. $33 = 21 + 8 + 3 + 1$).



Q.827 In the diagram three circles of equal size all pass through a point P. The triangle $\triangle ABC$ encloses the three circles, each side being tangential to two of them. Let I be the point inside the triangle equidistant from the three sides, this distance being r, and let O be the circumcentre of $\triangle ABC$, with $OA = OB = OC = R$. Prove that P lies on OI, and that $\dfrac{OP}{PI} = \dfrac{R}{r}$.

Q.828 Let $\triangle A_1B_1C_1$ be any given triangle. For $n = 2, 3, 4, \cdots$ let $\triangle A_nB_nC_n$ be the triangle inscribed in $\triangle A_{n-1}B_{n-1}C_{n-1}$ such that $C_nA_n \perp C_{n-1}A_{n-1}$, $A_nB_n \perp A_{n-1}B_{n-1}$, and $B_nC_n \perp B_{n-1}C_{n-1}$ (see figure).



(i) Show how to construct with ruler and compass the triangle $\triangle A_2B_2C_2$.

(ii) Show that there is a point P (independent of n) which lies on the circles having diameters $A_{n-1}A_n$, $B_{n-1}B_n$, and $C_{n-1}C_n$.