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THE DIVISOR GAME

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David Angell's recent article on addition games reminded me of a number game called the Divisor Game. The rules are fairly simple: A positive integer $N > 1$ is chosen. Two players, A and B, take turns selecting a divisor of N. The only stipulation is that no player can select a multiple of a previously selected divisor. The player who selects 1 loses the game.

For example, if $N = 72$, A may select any divisor of 72:

1 2 3 4 6 8 9 12 18 24 36 72.

Suppose that A chooses 6. Then B cannot choose multiples of 6:

1 2 3 4 ~~6~~ 8 9 ~~12~~ ~~18~~ ~~24~~ ~~36~~ ~~72~~.

If B chooses 4, then A must choose from

1 2 3 ~~4~~ ~~6~~ ~~8~~ 9 ~~12~~ ~~18~~ ~~24~~ ~~36~~ ~~72~~.

At this stage of the game, A should choose 9. B will take either 2 or 3, and A can choose the other. This forces B to select 1 and thereby lose. So choosing 4 was a mistake for B. B should have chosen 8 instead. Try to work out how B would play the rest of the game to win.

As it turns out, A should never lose this game, no matter what number N is used. Here's the argument that shows that A always has a winning strategy: If there is a winning strategy that begins with A selecting N, then obviously there is a winning strategy. Otherwise, if A selects N, then B can force A to lose. In other words, choosing N gives B a winning strategy. But A can adopt this winning strategy from the beginning. Whatever B would call in response to N, should be A's first call, etc ...

Before you run out and try to make lots of money off your friends playing this game, you have to work out two things. One is how to convince them to always let you go first.

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The other one is that the argument I gave is only an existence proof: it tells you absolutely nothing about how to find the right strategy. No one knows a method for devising such a strategy for a general N. However, with a little work, we can come up with some strategies for the numbers you'll typically encounter.

The first thing to notice is that if M and N have the same sequence of powers in their prime factorisations, then the games for M and N are virtually identical. For example, the game for $72 = 2^3 3^2$ would play the same as the game for any $N = p^3 q^2$, where p and q are different primes. To see this, consider the *divisor lattice* for $p^3 q^2$ (Figure 1). A divisor lattice is formed by arranging the divisors of N on different levels according to the sum of the powers in the divisor. The levels are ordered from the highest power sum to lowest. For $p^3 q^2$, the highest sum is 5. The lowest sum is always 0. Then line segments are drawn from each number to any multiple it has at the next highest level. Of course, the more prime divisors that N has, the more complicated the divisor lattice will become. The multiples of any number in the divisor lattice can be found by following the line segments up the lattice. For example, in Figure 1, the number $p^2 q^2$ can be reached from p, by going from p to pq to $p^2 q$ to $p^2 q^2$. (Clearly, there may be more than one route.) This tells us that we can think of this game as being played on the lattice: players alternate choosing vertices of the lattice. Calling a vertex eliminates from further play itself and the ones which can be reached by a rising path from the vertex. The player who calls the bottom vertex loses. Since numbers with the same sequence of powers in their prime factorisations have the same lattice structure, the corresponding games will be the same.

Now let's try to find A's winning strategy for a few simple games. From now on, p, q and r will represent distinct primes. If $N = p^t$, then the strategy is simple: choose p. B then has to choose 1, and the game ends.

If $N = p^t q$, then A should start with $p^t q$ itself. With the exception of the top and bottom levels, the lattice for $p^t q$ has two divisors on each level. Whenever B calls a divisor, A should simply respond by choosing the divisor on the same level. This forces B to a lower level with each selection, and eventually it forces B to the bottom level.

I'm going to run through the strategies for $p^t q^t, p^3 q^2$ and pqr. Try to work them out yourself, starting with $p^2 q^2$ as a warmup, before reading on. Here's a hint: except when

N is a power of a prime itself, A should never start out with a power of a prime. Such a choice leaves B with a fresh divisor lattice. For example, with $N = p^3q^2$, calling out p^2 leaves B the divisors of pq^2 from which to choose. That is like starting over with $N = pq^2$ and letting B go first. Then B should be able to win.

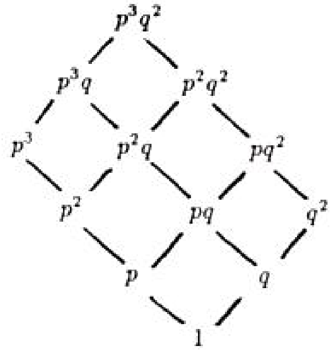

Figure 1: The divisor lattice for p^3q^2.

For $N = p^tq^t$, A should pick pq first. The only remaining choices for B are the powers of p and the powers of q. Whenever B picks a power of p, A can respond with the same power of q, and vice versa. This always forces B to lower levels, and eventually to the bottom.

For $N = p^3q^2$, the winning strategy begins with calling p^2q. If B chooses pq^2 or p^2, A should respond with the other. If B chooses p^3 or pq, A should respond with the other. If B selects q^2, A should respond with p^3. If B selects either p or q, A should take the other. With those responses, only two choices will be left on any eligible level, and A can close out any level B enters and force B to the bottom.

If $N = pqr$, then A should start with pqr. If B selects pq or r, then A should respond with the other. This leaves p, q and 1 for PB and the rest is clear. The other possibilities are the same by relabelling p, q and r.

You could also play a Subset Game: given a finite set X, players call out subsets of X, but sets containing previously called subsets are not allowed. The player forced to take the empty set loses. Is this substantially different from the Divisor Game for certain N's? (Think about what a subset lattice would look like.)

Try working out winning strategies for the Divisor Game with other choices for N. One you could start with is $pqrs$. If you work it out, you might like to try the game out on unsuspecting friends in 1995!