PROBLEM SECTION

You are invited to submit solutions to any or all of the following problems, accompanied by your name, school and year or form. Solutions of these problems will appear in the next issue of **Parabola**; your solution (s) may be used if they are received in time.

Q.957 Solve the three simultaneous equations

$$\frac{ab}{a+b}=\frac{1}{2}\quad,\quad\frac{bc}{b+c}=\frac{1}{3}\quad,\quad\frac{ac}{a+c}=\frac{1}{9}.$$

Q.958 By considering $(2+\sqrt{3})^n$ and $(2-\sqrt{3})^n$ show that the equation $x^2-3y^2=1$ has infinitely many integer solutions.

Q.959

(a) Suppose we are given 100 positive integers $x_1, x_2, \ldots, x_{100}$ between 1 and 100,000,000,000. Prove that for at least two of these numbers, say x_i and x_j, the sum of the digits of x_i is the sum of the digits of x_j (e.g, if $x_i = 1416$ and $x_j = 222222$ then the sums of the digits are the same).

(b) Suppose we have $n+1$ real numbers where $n \geq 2$. Prove that for two of these numbers a and b we must have

$$0<\frac{a-b}{1+ab}<\tan\frac{\pi}{n}.$$

Q.960 Suppose there are 500 students and 500 numbered lockers in a school. Initially all the lockers are open. The students line up and the first student closes every second locker, beginning with locker number 2. The next student then examines every third locker, beginning with locker number 3, and closes each locker if it is open whilst opening if it is closed. The remaining students continue this opening and shutting process. (The third student examines every fourth locker and so forth). Which lockers are open at the end of the procedure?

Q.961 Suppose a, b and c are roots of the equation $x^3-x^2-x-1=0$.

(a) Show that a, b and c are distinct.

(b) Show that

$$\frac{a^{1000000}-b^{1000000}}{a-b}+\frac{b^{1000000}-c^{1000000}}{b-c}+\frac{c^{1000000}-a^{1000000}}{c-a}$$

is an integer.

Q.962 Prove that

$$\frac{1\times 3\times 5\times\ldots\times 99}{2\times 4\times 6\ldots\times 100}=(1-\frac{1}{2})(1-\frac{1}{4})\ldots(1-\frac{1}{100})<\frac{1}{12}$$

[Hint: Try to discover a general result which can be proved by induction.]

Q.963 Starting from home I drive a certain distance due East. Then I turn 90° left and drive another distance, then turn left again and so on until I have driven $4n$ segments. Suppose the lengths of these segments are $1, 2, 3, \ldots, 4n$ kilometres but not necessarily in that order.

(a) What is the greatest distance I can eventually be from home?

(b) Can I end up at home?

Q.964 The Fibonacci numbers are defined by

$$F_{n+1} = F_n + F_{n-1} \quad \text{where } n \geq 1$$

subject to $F_0 = 0$ and $F_1 = 1$. Show that

$$F_n = {}^{n}C_0 + {}^{n-1}C_1 + {}^{n-2}C_2 + \ldots + {}^{n-k}C_k$$

where $k = \frac{n}{2}$ if n is even, $(n-1)/2$ if n is odd.

Interpret this equation in Pascal's triangle.

Q.965 A fraction is called a unit fraction if it has numerator 1.

(a) Show that every positive fraction can be written as a sum of distinct unit fractions.

(b) Given any two positive integers a and b, show that there exists a positive integer k such that ak can be written as a sum of distinct factors of bk.