**Parabola Volume 33, Issue 3 (1997)**

# THE MAGIC OF MOSCHOPOULOS

**Peter G. Brown.**

A **Magic Square** of order $n$ is an arrangment of the numbers $1, 2, ..., n^2$ into a square array in which we get the same sum whenever we add the numbers in any row, column or diagonal.

Perhaps the most famous magic square is the one which appeared in the top right hand corner of the painting *Melancholia* by Albrecht Dürer in 1514.

| | | | |
|---|---|---|---|
| 16 | 3 | 2 | 13 |
| 5 | 10 | 11 | 8 |
| 9 | 6 | 7 | 12 |
| 4 | 15 | 14 | 1 |

Not only is the sum of any row, column or diagonal equal to 34, but observe that the sum of the four corner numbers is also 34, and that the sum of the squares of the numbers in the top two rows equals the sum of the squares of the numbers in the bottom two rows. (There are several other such nice properties which I leave you to find.)

In a general magic square of order $n$, the *magic sum* $S$ to which the numbers in any row, column or diagonal add, can be found by noting that the sum of the entries of all the numbers in the square is equal to $n$ times the sum of each row which is $S$. We can then use the well-known formula for $1 + 2 + ... + k = \frac{1}{2}k(k + 1)$ to obtain

$$nS = 1 + 2 + 3 + ... + n^2 = \frac{1}{2}n^2(n^2 + 1)$$

and so

$$S = \frac{1}{2}n(n^2 + 1).$$

The Chinese are generally credited with the invention of Magic Squares, and in fact the earliest known magic square is called the *Lo Shu* which is a pictorial representation of the magic square below

| | | |
|---|---|---|
| 4 | 9 | 2 |
| 3 | 5 | 7 |
| 8 | 1 | 6 |

and makes its first appearance in the first century A.D. It is undoubtedly much older but there is no direct evidence as to how far back it goes, and claims of 2000 B.C. are probably far too extreme.

The idea of the magic square was transmitted to the Arabs from the Chinese in the eighth century and is discussed by Tabit ibn Korra (known for his formula for amicable numbers) in the early ninth. Lists of magic squares of all orders from 3 to 9 are displayed in the *Encyclopaedia*, (the *Rasā'il*), compiled about 990 by a group of Arabic scholars known as the 'brethren of purity' (the *Ikhwān al- ṣafa*). Despite all this, no general constructive methods appeared until slightly later. In 1225, Aḥmed al-Būnī showed how to construct magic squares using a simple bordering technique, but he may not have discovered the method himself.

Magic Squares were eventually introduced to the Western world in about 1315 by a Byzantine scholar named Manuel Moschopoulos. It is very unlikely that he worked out any of the methods for finding Magic Squares himself and there is some evidence that the methods explained by Moschopoulos may have been of Persian origin. (The work on Magic Squares by Moschopoulos has never been published in English[1] but there is an old French translation from last century.)

Moschopoulos was probably born about 1265, and was the nephew of Nicephoros Moschopoulos, who became bishop of Crete during the reign of the Emperor Andronikos II. In 1305 or 1306 he was involved in some kind of political plot resulting in his incarceration and disgrace, but we don't know much about the details. In his writings he gives a number of techniques for constructing magic squares of various types. I want to show you only one of the recipes he gives and attempt to explain where the idea came from which generates the recipe.

The *Lo Shu* magic square was created according to the scheme shown in Figure 0. The numbers from 1 to 9 are written in order in a diagonal fashion and a central square is removed, giving the basis of the magic square. The remaining numbers are then reflected from top to bottom and side to side to complete the square.

| | | | | |
|---|---|---|---|---|
| | | 1 | | |
| | 4 | | 2 | |
| 7 | | 5 | | 3 |
| | 8 | | 6 | |
| | | 9 | | |

$\longrightarrow$

| | | | | |
|---|---|---|---|---|
| | | 1 | | |
| | 4 | | 2 | |
| 7 | | 5 | | 3 |
| | 8 | | 6 | |
| | | 9 | | |

$\longrightarrow$

| | | |
|---|---|---|
| 4 | 9 | 2 |
| 3 | 5 | 7 |
| 8 | 1 | 6 |

Fig. 0

The idea of this construction generalises to produce magic squares with an odd number of sides. (Note that there are other Magic Squares with an odd number of sides which do not arise from this method.) I will illustrate for the magic square of side 5.

Draw up the 9 sided square grid as shown, (Fig 1a.) and place the numbers 1 to 25 in standard order as shown. From this we extract the central square with entries and spacings as shown (Fig. 1b.)

[1]The present writer has recently submitted a translation from the Greek for publication to a Journal dealing with the History of Mathematics.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 1 | | | | |
| | | | 6 | | 2 | | | |
| | | 11 | | 7 | | 3 | | |
| | 16 | | 12 | | 8 | | 4 | |
| 21 | | 17 | | 13 | | 9 | | 5 |
| | 22 | | 18 | | 14 | | 10 | |
| | | 23 | | 19 | | 15 | | |
| | | | 24 | | 20 | | | |
| | | | | 25 | | | | |

Fig. 1a.

| | | | | |
|---|---|---|---|---|
| 11 | | 7 | | 3 |
| | 12 | | 8 | |
| 17 | | 13 | | 9 |
| | 18 | | 14 | |
| 23 | | 19 | | 15 |

Fig. 1b.

| | | | | |
|---|---|---|---|---|
| 11 | 24 | 7 | 20 | 3 |
| 4 | 12 | 25 | 8 | 16 |
| 17 | 5 | 13 | 21 | 9 |
| 10 | 18 | 1 | 14 | 22 |
| 23 | 6 | 19 | 2 | 15 |

Fig. 1c.

The remaining numbers are transposed from the top to the bottom and side to side to obtain the final square. (Fig. 1c.)

Now this is NOT the recipe given by Moschopoulos, but I believe that is was the underlying method which led Moschopoulos, or rather those from whom he got the recipe, to devise the recipe in the first place. This is typical of what can happen in mathematics when someone finds a simple recipe or method for doing something. The original reasoning which led to the method gets forgotten and people only remember the 'formula' or method and forget the ideas which led to it. (This is clearly NOT a good thing!)

Moschopoulos' method for constructing the same squares is as follows. Suppose we wish to construct a $3 \times 3$ square. We draw up the grid and place a 1 in the middle cell along the bottom row. The basic rule for placing the numbers is to move **down one** and **to the right one**, (i.e. move diagonally to the right) . In this case we end up out of the grid so we move to the opposite side (i.e. the top) and continue to apply the rule.

| | | |
|---|---|---|
| 4 | | 2 |
| 3 | | |
| | 1 | |

Thus the 2 ends up in the top right cell. Moving 'down one across one', the 3 ends up out of the cell and so we go to the opposite side and place it as shown. Now, says Moschopoulos, when we reach a number which is a multiple of the size of the square, the next number is placed directly above the multiple we have reached, and so the 4 is placed directly above the 3 as shown. You can now finish off the square following the same patterns, to get

| | | |
|---|---|---|
| 4 | 9 | 2 |
| 3 | 5 | 7 |
| 8 | 1 | 6 |

which is the Lo Shu square.

For the $5 \times 5$ odd square, we follow the same pattern except that we place the initial 1 in the middle cell of the second line from the bottom, and for the $7 \times 7$ square,we choose the middle cell of the third row from the bottom and so on. For the $n = 2r + 1$ side square we place the 1 in the middle of the $r$th row from the bottom.

| | | | | |
|---|---|---|---|---|
| 11 | 24 | 7 | 20 | 3 |
| 4 | 12 | 25 | 8 | 16 |
| 17 | 5 | 13 | 21 | 9 |
| 10 | 18 | 1 | 14 | 22 |
| 23 | 6 | 19 | 2 | 15 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 22 | 47 | 16 | 41 | 10 | 35 | 4 |
| 5 | 23 | 48 | 17 | 42 | 11 | 29 |
| 30 | 6 | 24 | 49 | 18 | 36 | 12 |
| 13 | 31 | 7 | 25 | 43 | 19 | 37 |
| 38 | 14 | 32 | 1 | 26 | 44 | 20 |
| 21 | 39 | 8 | 33 | 2 | 27 | 45 |
| 46 | 15 | 40 | 9 | 34 | 3 | 28 |

You should try making up the $9 \times 9$ square using this method.

Clearly the first method was developed to actually contruct the first few squares and then the 'simpler' method was developed which gave the same squares but without having to draw the larger grids.

If you have read this far, you may now be happy that you can build your own magic squares. On the other hand you may want to know exactly why the first method works. If so read on!

If you look carefully at the examples given (and try a few yourself) you should be able to see that the sum of the 'main diagonal' (i.e. the diagonal from the top left to the bottom right) of an $n$ sided square will be the sum of the $n$ consecutive integers starting from $\frac{n^2-n+2}{2}$, which gives the sum

$$\left(\frac{n^2-n+2}{2}\right) + \left(\frac{n^2-n+2}{2}+1\right) + .... + \left(\frac{n^2-n+2}{2}+(n-1)\right)$$

which is simply an Arithmetic series and can be summed using the usual formula to give $\frac{n}{2}(n^2+1)$.

The sum of the numbers on the diagonal, from the top right to the bottom left, starts with $\frac{n+1}{2}$ and increases by $n$ each time, so we are summing

$$\left(\frac{n+1}{2}\right) + \left(\frac{n+1}{2} + n\right) + \ldots + \left(\frac{n+1}{2} + n(n-1)\right).$$

This is also an Arithmetic series and again you can show it sums to $\frac{n}{2}(n^2+1)$.

The columns are a little more difficult to see. Recall that the square of side $n$ was (in the first method) constructed using a grid of size $2n-1$. I will illustrate the idea using the $5 \times 5$ square.

| | | | | |
|---|---|---|---|---|
| 11 | 24 | 7 | 20 | 3 |
| 4 | 12 | 25 | 8 | 16 |
| 17 | 5 | 13 | 21 | 9 |
| 10 | 18 | 1 | 14 | 22 |
| 23 | 6 | 19 | 2 | 15 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 1 | | | | |
| | | | 6 | | 2 | | | |
| | | 11 | | 7 | | 3 | | |
| | 16 | | 12 | | 8 | | 4 | |
| 21 | | 17 | | 13 | | 9 | | 5 |
| | 22 | | 18 | | 14 | | 10 | |
| | | 23 | | 19 | | 15 | | |
| | | | 24 | | 20 | | | |
| | | | | 25 | | | | |

Now adding, for example, the first column of the 5 sided square is the same as adding the entries in the 3rd column of the 9 sided square plus the entries in the 8th column of that square. The second column of the magic square is simply the sum of the 4th and 10th columns of the larger square and so on.

In general the sum of the general column is

$$\sum_{k=0}^{j-1}[1 + (n-j)n + (n+1)k] + \sum_{k=0}^{n-j-1}[(j+1) + (n+1)k]$$

which, as expected, simplifies to $\frac{n}{2}(n^2+1)$. The row sum can be similarly shown to be $\frac{n}{2}(n^2+1)$.

There are other (similar but slightly simpler) algorithms for producing other magic squares of odd order, and also rules for moving lines and columns around to get new squares from old ones. Moschopoulos does give a second method for producing squares of odd order and also a method for producing magic squares whose order is a power of 2. The more general problem of finding magic squares of even order is not covered in Moschopoulos and is in fact somewhat more difficult. The interested reader can consult the 'standard' book on the subject *Magic Squares and Cubes* by W.S. Andrewes, (Dover Press), but note that some of his comments on the history of magic squares are far-fetched and incorrect. Interestingly the methods given by Moschopoulos are NOT in his book (although similar methods are).

As far as I am aware magic squares have no practical value although they were believed in the Middle Ages to have 'magical powers'. Nonetheless, they are fun to construct and there are many interesting combinatorial questions associated with them.

For example, how many magic squares of given order are there? As far as I am aware this problem has not yet been fully solved.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 16 | 50 | 61 | 3 | 48 | 18 | 29 | 35 |
| 53 | 11 | 8 | 58 | 21 | 43 | 40 | 26 |
| 4 | 62 | 49 | 15 | 36 | 30 | 17 | 47 |
| 57 | 7 | 12 | 54 | 25 | 39 | 44 | 22 |
| 9 | 55 | 60 | 6 | 41 | 23 | 28 | 38 |
| 52 | 14 | 1 | 63 | 20 | 46 | 33 | 31 |
| 5 | 59 | 56 | 10 | 37 | 27 | 24 | 42 |
| 64 | 2 | 13 | 51 | 32 | 34 | 45 | 19 |

An eighth order magic square .