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'REGULAR' POLYGONS

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The article "From the Archives... Impossible Constructions" (Anonymous, 1999. Parabola, 35 (1), pp. 12-18) reminded me of the method which a friend, who is a Draftsman, uses to construct what he thinks are regular figures in his drawings. Here are his instructions to construct a "regular" polygon with n sides, using only a straight edge and collapsible pair of compasses. In diagram 1, an example is shown for $n = 5$.

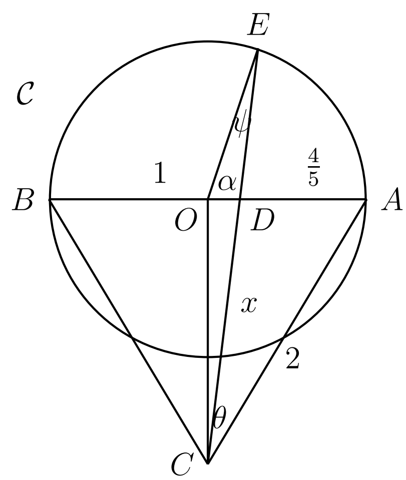

Diagram 1(a). The Draughtsman's Construction of a 'regular' pentagon.

Let $\mathcal{C}$ be a circle, centre O, radius 1.

1. Draw the diameter AB, and divide it into n equal subintervals. This can be readily achieved by drawing another line which is divided into n subintervals, and then constructing a set of parallel lines which intersect AB.

2. Choose the point D to be the point which divides the interval AB internally in the ratio $2 : n - 2$. In the example, AD is $\frac{2}{5}$ of AB, that is $AD = \frac{4}{5}$.

3. Construct the equilateral triangle ΔABC.

4. Draw the line CD extended to meet the circle $\mathcal{C}$ at E.

AE will form one side of the "regular" polygon with n sides. You will see in the example of a "regular" pentagon in diagram 1(b) that the resulting figure does appear to be regular.

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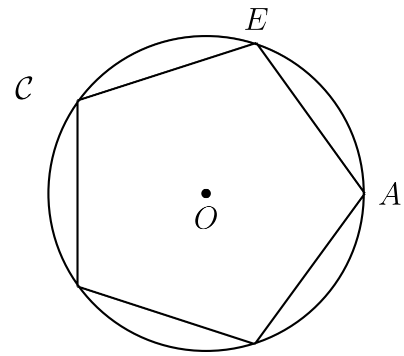


Diagram 1(b). A "regular" pentagon.

Now the article "Impossible Constructions" clearly points out that the Draftsman is making a very broad and bold claim. It is impossible, for example, to construct a regular heptagon (7 sided figure) using straight edge and pair of collapsible compasses alone. So some questions immediately and naturally arise. How accurate is the Draftsman's Construction? Who first discovered the Draftsman's Construction? And in what context did it appear?

I can only go a small way towards answering the first question. Let us consider the Draftsman's Construction of an n-gon. Since the radius of the circle is 1,

$$AC = 2, \quad AD = \frac{4}{n}, \quad OD = \frac{(n-4)}{n}.$$

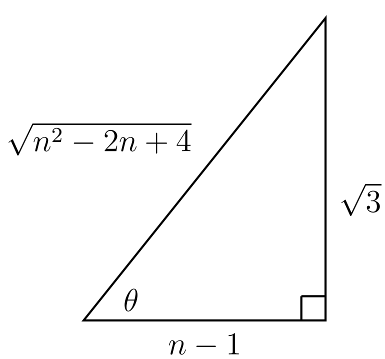
Let $CD = x,\ \angle ACD = \theta,\ \angle OED = \psi$ and $\angle DOE = \alpha,$ as shown in the example of the pentagon in diagram 1.

In $\Delta ODC, OD = \frac{1}{n}$, and $OC = \sqrt{3}$ because it is the altitude of an equilateral triangle. Therefore, using the Cosine Rule in ΔACD we have

$$x = \frac{2}{n}\sqrt{n^2 - 2n + 4}.$$

One application of the Sine Rule in ΔACD gives

$$\sin\theta = \frac{\sqrt{3}}{\sqrt{n^2 - 2n + 4}}.$$



Sketch 1.

Another application of the Sine Rule in ΔODE, along with the expansion for $\sin(120°-\theta)$ and the value for $\cos\theta$ obtained from the information in sketch 1, shows that

$$\sin\psi = \frac{(n-4)\sqrt{3}}{2\sqrt{n^2-2n+4}}.$$

Since ΔABC is equilateral, $\alpha = 60° + \theta - \psi$.
Of course, we would really like α to equal $2\pi/n$ (or $\frac{360°}{n}$).

n	$\frac{360°}{n}$	θ	ψ	α	Error
3	120°	40°54′	−19°06′	120°	0
4	90°	30°	0	90°	0
5	72°	23°25′	11°28′	71°57′	−3′
6	60°	19°06′	19°06′	60°	0
7	51°26′	16°06′	24°35′	51°31′	5′
8	45°	13°54′	28°43′	45°11′	11′
9	40°	12°13′	31°56′	40°17′	17′
10	36°	10°54′	34°32′	36°21′	21′
11	32°44′	9°50′	36°41′	33°09′	25′
12	30°	8°57′	38°28′	30°28′	28′

Table 1. Angles are correct to the nearest minute where applicable.

Now the functions for θ and ψ, and therefore also α, are eminently suitable for keying into a programmable calculator (and if you are lucky it will have a graphics mode). Table 1 above gives the output for θ, ψ and α for $n = 3$ to 12. As you would expect, the graph for θ decreases steadily and approaches (asymptotically) 0, and the graph for ψ increases steadily and approaches (asymptotically) $60°$. By "error" is meant the difference between α and the hoped-for value of α. The small size of the error, at least in absolute terms, immediately strikes the eye. My calculator indicated a maximum value of the function $E(n) = \alpha - \frac{2\pi}{n}$ (or $\alpha - \frac{360°}{n}$) at $n = 22$ of 0.01 of a radian, or about 38 minutes. If the relative error is expressed as a percentage of $2\pi/n$, another surprise becomes evident. For small n ($n = 3$ to 10), the percentage error is less than 1%. As n increases, the percentage error slowly increases, but appears to approach a limit of about 10%.

So the Draftsman's Construction of a "regular" polygon is remarkably accurate. However, this exploration of the behaviour of α, x and y, their error functions and their graphs, has given no indication of what mathematical motivation may have lain behind the development of the Draftsman's Construction. Was it, perhaps, just wishful thinking and a happy accident?

Here are two constructions of the real, fair dinkum, honest to goodness regular pentagon, based on $\cos\frac{\pi}{5} = (1+\sqrt{5})/4$.

In diagram 2, $OF \perp AB$ and C is the midpoint of OF. The bisector of $\angle OCA$ meets AB at D and $DE \perp AB$.

(See H. Coxeter, 1969. Introduction to Geometry, 2nd Ed., New York ; John Wiley and Sons. p. 27.)

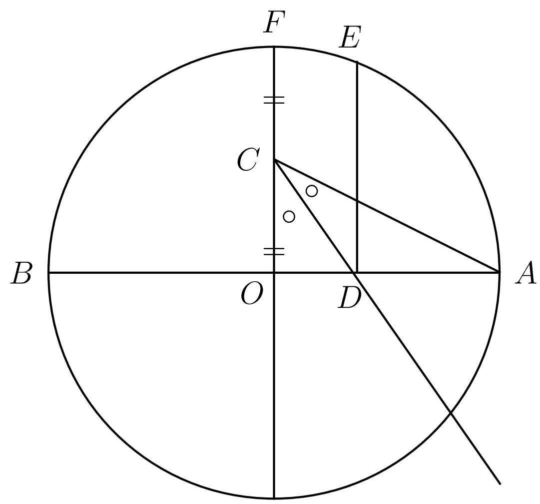


Diagram 2.

In diagram 3, $OF \perp AB$ and C is the midpoint of OB. Also $CD = CF$ and $BE = BD$.

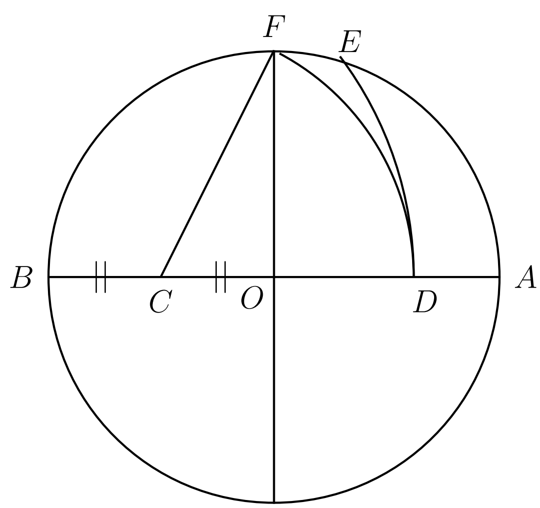


Diagram 3.

In both cases, AE will form one side of a regular pentagon inscribed in the circle.

EDITORIAL NOTE.

In the above article, Peter Merrotsy posed the question as to why the Draftsman's Construction was so accurate. Mike Hirschhorn, who is a lecturer at UNSW and on the editorial board for Parabola, gave the following explanation as to the accuracy of Diagram 1.

Using the diagram as before, we suppose that E is placed in exactly the correct spot to make AE a side of the true n-gon. (In the diagram, $n = 5$.) We then ask the question, where should we place D so that ED passes through the point C, where ΔABC is an equilateral triangle? Let the $\angle EOA = \frac{2\pi}{n}$ and consider the centre of the circle C to be the origin O on the Cartesian Plane. Also set the point E to be $(\cos\frac{2\pi}{n}, \sin\frac{2\pi}{n})$ and write the point C as $(0, -\sqrt{3})$. By finding the equation of the line through EC and setting $y = 0$, we find the x coordinate (abscissa) of D to be

$$x = \frac{\sqrt{3}\cos\frac{2\pi}{n}}{\sin\frac{2\pi}{n} + \sqrt{3}}.$$

Mike then considered the ratio

$$BD : DA = 1 + x : 1 - x$$

and found (after some work) that

$$\frac{1+x}{1-x} = \frac{\sin(\pi/n + \pi/3).\cos\pi/n}{\cos(\pi/n - \pi/3).\sin\pi/n} = f(n).$$

Surprisingly, the graph of this function $f(n)$ is almost a straight line for $n > 2$. (Try using your graphics calculator to sketch this function.) The function $f(n)$ is very close to $(n-2)/2$, and in fact as n becomes large the ratio $f(n) : (n-2)/2$ approaches $2\sqrt{3}/\pi$, which is approximately 1.1. Note that this error, too, is about 10%. This explains why the construction is so accurate.

Editor.