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SOLUTIONS TO PROBLEMS 1092-1097

Q1092 Carl Frederick Gauss, born in Braunschweig in 1777 and who died in Göttingen in 1855, was one of the greatest mathematicians of all time. Assuming that each letter in the following sum represents a different digit find all possible solutions for A, C, G, L, R, S and U.

$$\begin{array}{rrrrr} & 1 & 7 & 7 & 7 \\ & 1 & 8 & 5 & 5 \\ & C & A & R & L \\ \hline G & A & U & S & S \\ \hline \end{array}$$

ANS. There are only a finite number of solutions; indeed the number is at most $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 604,800$.
The value of L determines the value of both S and R, as follows:

L	S	R
0	2	9
1	3	0
2	4	1
3	5	2
4	6	3
5	7	4
6	8	5
7	9	6
8	0	6
9	1	7

Next, since $10G + A \leq 2 + C + 2,\ G = 1$. Now C must be at least 6 since the carry from third to the fourth column is at most 2. Consider

$$\begin{array}{rrrrr} & 1 & 7 & 7 & 7 \\ + & 1 & 8 & 5 & 5 \\ + & C & A & R & L \\ \hline 1 & A & U & S & S \end{array}$$

Since $C \leq 9$ and the carry from the third to the fourth column is at most $2,\ A$ is either $0, 1, 2, 3$. If the carry is 2 then A must be 3, and then C is 9. If the carry is 1 then $(C, A) = (7, 0), (8, 1)$ or $(9, 2)$. But $A \neq 1$: hence $(C, A) = (9, 3), (9, 2)$ or $(7, 0)$. We have reduced the problem to exactly 30 cases which have to be (essentially) individually checked yielding exactly 3 solutions

$$\begin{array}{rrrrr} & 1 & 7 & 7 & 7 \\ & 1 & 8 & 5 & 5 \\ & 9 & 2 & 3 & 4 \\ \hline 1 & 2 & 8 & 6 & 6 \end{array} \quad \begin{array}{rrrrr} & 1 & 7 & 7 & 7 \\ & 1 & 8 & 5 & 5 \\ & 9 & 2 & 4 & 5 \\ \hline 1 & 2 & 8 & 7 & 7 \end{array} \quad \begin{array}{rrrrr} & 1 & 7 & 7 & 7 \\ & 1 & 8 & 5 & 5 \\ & 7 & 0 & 2 & 3 \\ \hline 1 & 0 & 6 & 5 & 5 \end{array}$$

Q1093 Find a positive integer, the first digit of which is a 1 and which has the property that the number is tripled if the 1 is moved to the end of the number. Can you find all such numbers?

ANS. Suppose the number is $\ldots a_7\, a_6\, a_5\, a_4\, a_3\, a_2\, a_1$. Then

$$\begin{array}{cccccccc} \ldots & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \\ & & & & & & & 3 \\ \hline \ldots a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & 1 \end{array}$$

We now deduce $a_1 = 7, a_2 = 5, a_3 = 8, a_4 = 2$ and $a_5 = 4$. So consider

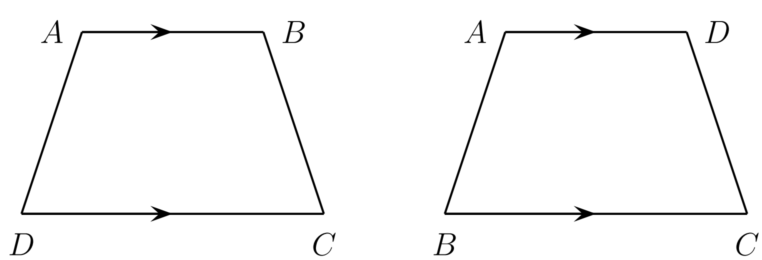
$$\begin{array}{ccccccc} \ldots a_7 & a_6 & 4 & 2 & 8 & 5 & 7 \\ & & & & & & 3 \\ \hline \ldots a_6 & 4 & 2 & 8 & 5 & 7 & 1 \end{array}$$

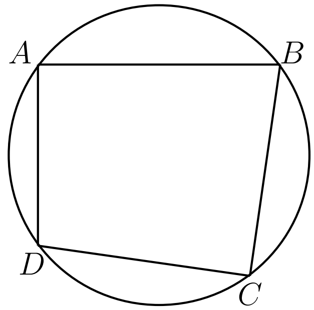
We next deduce $a_6 = 1$ so either the number is exactly $1\,4\,2\,8\,5\,7$ or it is longer than 6 digits. Repeating the above argument we show the number is $1\,4\,2\,8\,5\,7\,1\,4\,2\,8\,5\,7$ or longer. Clearly every number with this property has length $6n$ and consists of the block $1\,4\,2\,8\,5\,7$ repeated n times.

Q1094 The angles A, B, C and D of a convex quadrilateral satisfy the equation

$$\cos A + \cos B + \cos C + \cos D = 0.$$

Prove that $ABCD$ either is a trapezium or is cyclic.
ANS.





The convex quadrilateral $ABCD$ is a trapezium if and only if $A + D = B + C = 180°$ or $A + B = C + D = 180°$. $ABCD$ is a cyclic quadrilateral if and only if $A + C = B + D = 180°$, as in the diagrams above.

To solve the problem we need some trigonometric identities:

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

so

$$\cos(x + y) + \cos(x - y) = 2 \cos x \cos y$$

If we set $x + y = A,\ x - y = B$ then $x = (A + B)/2$ and $y = (A - B)/2.$ Similarly if $x + y = C$ and $x - y = D$ then $x = (C + D)/2$ and $y = (C - D)/2$. Thus

$$\cos A + \cos B + \cos C + \cos D = 0$$

becomes

$$2 \cos \frac{A + B}{2} \cos \frac{A - B}{2} + 2 \cos \frac{C + D}{2} \cos \frac{C - D}{2} = 0$$

Now $A + B + C + D = 360°$ so $\dfrac{A + B}{2} = 180° - \dfrac{C + D}{2}$ and $\cos \dfrac{A + B}{2} = -\cos \dfrac{C + D}{2}$ and we now have

$$\cos \frac{A + B}{2} = 0 \text{ or } \cos \frac{A - B}{2} = \cos \frac{C - D}{2}$$

Finally

$$\frac{A + B}{2} = 90° \text{ or } \frac{A - B}{2} = \pm \frac{C - D}{2}$$

so $A + B = 180°$ or $A + D = B + C$ or $A + C = B + D$, as required.

Q1095 Consider the triangle of numbers in the diagram below. Each number is the sum of three numbers in the row above: the number above it and the numbers immediately to the left and the right of that number. (A blank is treated as a zero.) Prove that from the third row onwards, every row contains at least one even number. Compare the situation with Pascal's triangle.

				1				
			1	1	1			
		1	2	3	2	1		
	1	3	6	7	6	3	1	
.	.	.	.	.	.	.	.	.

ANS.

						1							0
					1	1	1						1
				1	2	3	2	1					2
			1	3	6	7	6	3	1				3
		1	4	10	16	19	16	10	4	1			4
	1	5	15	30	45	51	45	30	15	5	1		5
1	6	21	50	90	126	141	126	90	50	21	6	1	6

A little calculation leads to the above table. Let us number the rows $0, 1, 2, \ldots$ as above. Clearly the second entry in the n'th row is n. So every even row contains an even number, except the 0'th.

The 5th row begins with 3 odd numbers but the 4th number is even so we are led to consider the first four numbers of every row but simply recording whether they are odd or even.

						O		0
					O	O	O	1
				O	E	O	E	2
			O	O	E	O		3
		O	E	E	E			4
	O	O	O	E				5
O	E	O	E					6

The pattern in the second row is the same as in the sixth row hence the patterns repeat every four rows. Each row from 2 to 5 contains an even number, hence so does every row using periodicity.

NOTE: It is possible to show that the second entry in each row is n, the third entry is $n(n+1)/2$ and the fourth entry is $n(n+4)(n-1)/6$.
If $n \equiv 0, 1$ or $2 \pmod 4$ then $n(n+4)(n-1)/6$ is even. Similarly $n(n+1)/2$ is even if $n \equiv 0$ or $3 \pmod 4$. This provides a different proof of the result.

Q1096 Find all integers a, b and c for which

$$(x-a)(x-10)+1=(x+b)(x+c) \quad \text{for all } x.$$

ANS. If $(x-a)(x-10)+1 \equiv (x+b)(x+c)$ for all real x then putting $x=10$ we obtain $1=(10+b)(10+c)$. Thus there are only two possibilities, $10+b=10+c=1$ or

$10 + b = 10 + c = -1$. We also know $10a + 1 = bc$ and $10 + a = -b - c$ by equating the coefficient of x and the constant term in the equality. So the solutions are

$$a = 8,\ b = -9,\ c = -9 \quad \text{or} \quad a = 12,\ b = -11,\ c = -11.$$

Q1097 The first n prime numbers $2, 3, 5, \ldots, p_n$ are partitioned into two disjoint sets, A and B. The primes in A are $a_1,\ a_2,\ \ldots, a_h$, and the primes in B are $b_1,\ b_2,\ \ldots,\ b_k$, where $n = h + k$. The two products

$$C = a_1^{\alpha_1}\, a_2^{\alpha_2}\ \cdots\ a_h^{\alpha_h} = \prod_{i=1}^{h} a_i^{\alpha_i} \qquad \text{and} \qquad D = \prod_{i=1}^{k} b_i^{\beta_i}$$

are formed, where the α_i and the β_i are any positive integers. If d divides $C - D$, prove that either $d = 1$ or $d > p_n$.

ANS. Suppose $C - D = dE,\ d \neq 1$ and p is a prime divisor of d.
Suppose further that $p \leq p_n$ then $p = a_i$ some i or $p = b_j$ some j.
If $p = a_i$ then $p|d$ and $p|C$. But $D = C - dE$ so $p|D$ as well, which is not possible. Similarly if $p = b_j$ then $p|d$ and $p|D$ which implies $p|C = D + dE$.
Hence $p > p_n$ and $d > p_n$, as required.