Parabola Volume 37, Issue 3 (2001)

SHOCKING TRAFFIC

Bruce Henry[1]

I drive about an hour each way to work each day through heavy commuter traffic. Some days I can do the trip in under forty minutes whereas on other days it may be closer to two hours. On some of these slow trips you might see a car broken down which is obstructing one of the lanes of traffic or an accident that has brought things to a stand still. On other slow trips you may see nothing at all by way of explanation. The typical scenario in this latter case is that you seem to be flowing along nicely with the traffic and then all of a sudden you come to very slow traffic or a sudden halt, and then just as suddenly you are on your way again, but maybe in faster or slower traffic conditions than originally. If this has happened to you then its more than likely that you have been hit by a shock wave.

One of the ways to model traffic flow is based on a fluid flow analogy - a long column of closely spaced cars on a roadway is analogous to a stream of fluid in a channel. We begin with a few definitions. First we define the **traffic density** m. This is the number of cars per kilometre in a stream of traffic. Let u denote the **average speed** of the stream of traffic. The final definition that we require is the **flux** or rate of flow of the traffic which we represent by q. This measures the number of cars per hour in the stream that move past a given fixed position. Our three variables m, u and q are simply related by the formula

$$q = mu.$$

It is easy to verify that this formula is dimensionally correct;

$$\text{\# cars per hour} = \text{\# of cars per kilometre} \times \text{\# kilometres per hour}.$$

Note that each of our dynamical variables is itself a function of two variables; position x, and time t. Thus we write $q = q(x,t), m = m(x,t), u = u(x,t)$.

One of the fundamental equations in fluid dynamics is the equation of conservation of mass. If the flux and the density are smooth functions of position and time then this equation takes the form of a differential equation. But in some cases m is not a smooth function of position and time. For example a fairly open stream of traffic with density m_1 and flux q_1 may approach a tightly bunched stream of traffic with density m_2 and flux q_2. The traffic density will then change abruptly where the two streams meet and

[1]Dr Bruce Henry is a Senior Lecturer in the Department of Applied Mathematics at UNSW, where traffic flow problems and other real world problems feature in undergraduate courses.

the conservation of mass condition becomes instead represented by the equation

$$V = \frac{q_2 - q_1}{m_2 - m_1}.$$

The new quantity introduced on the left hand side of this equation has the dimensions of velocity,

$$\frac{\text{\# cars per hour}}{\text{\# cars per kilometre}} = \text{\# kilometres per hour}.$$

It represents the velocity of the interface where the two streams meet. The position of the interface is not fixed in general but is in motion. The motion is called a **shock wave** and V is called the **shock velocity**.

We are now ready to investigate the sort of scenario that might lead to the shocking traffic conditions referred to earlier. Just to make things a little less abstract we'll put in a few numbers. Consider a stream of traffic with 30 cars per kilometre travelling at 100 kilometres per hour that is suddenly blocked by an accident. The traffic density in the stationary queue held up by the accident is 180 cars per kilometre. Suppose that the accident is cleared off the road after 30 minutes and then the traffic moves off again but at a slightly more subdued speed of 80 kilometres per hour but with a higher density of 60 cars per kilometre. What will a driver experience if they are initially 110 kilometres from the accident site? We'll come back to this question in a moment after we have done a bit of preliminary analysis.

Let q_1, m_1, u_1 denote the traffic conditions in the initial stream of incoming traffic, let q_2, m_2, u_2 denote the traffic conditions in the stationary queue and let q_3, m_3, u_3 denote the traffic conditions in the stream of traffic that moves off after the accident is cleared. We can immediately write down; $m_1 = 30, u_1 = 100, \Rightarrow q_1 = 3000$; $m_2 = 180, u_2 = 0, \Rightarrow q_2 = 0$; $m_3 = 60, u_3 = 80, \Rightarrow q_3 = 4800$. The accident brings traffic to a halt and the traffic starts to queue up with the interface between the end of the stationary queue and the incoming traffic moving with a shock velocity of

$$V_A = \frac{q_1 - q_2}{m_1 - m_2} = -20\text{kph}.$$

The minus sign tells us that the shock is moving back into the incoming traffic as expected. When the accident if finally cleared after 30 minutes this shock will have travelled a distance of $L = V_A t = 20(1/2) = 10$ kilometres.

Now the traffic starts to move off again. Starting with cars at the head of the queue near the accident site. This change in traffic conditions after the accident is cleared sends a second shock through the system with velocity

$$V_B = \frac{q_2 - q_3}{m_2 - m_3} = -40\text{kph}.$$

The second shock moves in the same direction as the first, in the direction opposite to the traffic flow. Because it is going faster than the first shock it will eventually catch up with the first shock and at this point the stationary queue will be cleared. To work out the time it takes for the second shock to catch up with the first shock we solve the coupled equations

$$\begin{aligned} x_A &= -20t - 10 \quad \text{first shock position} \\ x_B &= -40t \quad \text{second shock position} \\ x_A &= x_B. \end{aligned}$$

This yields the time $t = 1/2$ an hour. Note that $x_A = x_B = -20$ kilometres at this time.

When the stationary queue is cleared we get a third shock wave in the system with speed

$$V_C = \frac{q_1 - q_3}{m_1 - m_3} = 60\text{kph}$$

now moving in the direction of the traffic.

Now consider our driver who was 110 kilometres from the accident when it happened. At a later time t the driver is at position

$$x = 100t - 110.$$

In particular after $t = 1/2$ an hour the driver is at $x = -60$ which is well short of the end of the stationary queue which is at -20 at this time. Which shock will the driver encounter first, the first shock or the second? Recall that after half an hour the first shock is at $x = -10$ and the second is at $x = 0$. We thus need to solve for the minimum time t in either

$$100t - 60 = -10 + V_A t$$

or

$$100t - 60 = V_B t.$$

The first time yields $t = 25$ minutes which is about one minute less than the second time. So the first shock, manifest by the stationary queue, will reach the driver first and then about a minute later a second shock, manifest by the clearing traffic after the accident, will reach the driver.

From the driver's perspective it would appear that they are enjoying good traffic when suddenly the traffic comes to an abrupt halt for almost a minute and then just as suddenly it moves off again out at a slightly reduced speed. The driver can't find any reason for this strange behaviour. After all, the driver is some twenty kilometres from where the accident occurred when this strange behaviour occurs and the accident has already been cleared off in any case, making it unlikely that the driver would notice anything when they finally do pass the accident site.

Photo courtesy: Catrine Larsson