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MATHEMATICAL MODELLING AND THE TIM-TAM COMPETITION

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On 26 October 2001 a large block of ice was unveiled inside an ice fridge at Darling Harbour in Sydney to launch the start of the "World's Coolest Tim-Tam" promotion. A packet of Tim-Tam biscuits was located near the centre of the block of ice and the promotion involved a 'guessing competition' to estimate how long it would take the ice to melt and release the Tim-Tam packet. With a $1,000,000 prize money on offer the competition attracted large numbers of entries and large numbers of visits to the competition WEB site `http://www.timtam.com.au` which featured up to date webcam images of the melting ice.

The competition appealed to me as a possible Mathematical Modelling problem. However after a close inspection of the competition rules I realized that it wasn't going to be easy. To win it was necessary to state the time to the nearest second at which the first part of the Tim-Tam packet would touch the floor. The competition WEB site did however reveal a few details, enough to entice a mathematical modeller: The Tim Tam packet weighs 200g. The block of ice is approximately 6 foot wide by 6 foot deep and 12 foot high (i.e., 1.8m $\times$ 1.8m $\times$ 3.6m) with an approximate weight of 15 tonnes. The fridge unit is fully enclosed with the temperature set at approximately -20C. On Sunday 11 November 2001 at 12 midnight EST, the refrigeration device will be switched off, and a heating device set at approximately 20C will be activated for the duration of the meltdown. Up to 200 entries per person.

So how to begin the modelling? Perhaps the first step these days is to turn to your favourite search engine (e.g., `GOOGLE`) and find out what you can. I tried a search for entries using the keywords "ice melting" and found all sorts of stuff about global warming and ice melting products. I searched again excluding global warming and ice melting products but I still didn't find anything useful. It was then off to a more serious search through the University of New South Wales online databases. The Science Citation Index is excellent. Lots of nice technical articles about almost anything; but not yet Tim-Tam packets encased in blocks of ice.

Time for thought. The higher the temperature difference between the surface of the ice and the surrounding air, the faster it will melt. The larger the surface area of a given quantity of ice (the more of it that is exposed to the air), the faster it will melt. These two elementary considerations *suggest* a primitive mathematical model in which the rate of melting is proportional to the surface area of the ice and the ice-air temperature

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difference. This can be expressed as a mathematical equation as follows:

$$\frac{dV}{dt} = kA(t)(\theta_{air}(t) - \theta_{ice}(t)). \tag{1}$$

In this equation, $V(t)$ is the volume of ice at time t; $A(t)$ is the surface area of ice at time t; $\theta_{air}(t)$ is the air temperature at time t; $\theta_{ice}(t)$ is the ice temperature at time t; and k is a proportionality constant (yet to be determined). If the surrounding air and the surface of the ice are at constant temperatures then our equation simplifies further to:

$$\frac{dV}{dt} = kA(t)\Delta\theta \tag{2}$$

where $\Delta\theta$ is now the constant temperature difference between the air and ice (twenty degrees Celsius in the case of the Tim-Tam competition).

Equation (2) is a first order differential equation which would be easy to solve if we had an explicit expression for $A(t)$. Here we do not have an explicit expression for $A(t)$ any more than we do for $V(t)$, the function that we are trying to evaluate. However suppose that the block of ice was in the shape of a sphere, then it would be reasonable to suppose that it would continue to maintain this isotropic shape as it melted. Thus for all times t, eliminating r in $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$ we would have

$$r = (3V)/4\pi)^{1/3} = (4\pi)^{-1/3}3^{1/3}V^{1/3}$$

$$A(t) = 4\pi r^2 = (4\pi)^{\frac{1}{3}}3^{\frac{2}{3}}V^{\frac{2}{3}}(t). \tag{3}$$

More generally if the block of ice maintains its shape as it melts then

$$A(t) = \alpha V^{\frac{2}{3}}(t) \tag{4}$$

where the constant of proportionality α depends on the shape; $\alpha = (4\pi)^{\frac{1}{3}}3^{\frac{2}{3}}$ for a sphere and $\alpha = 6$ for a cube. With this shape preserving *assumption* [2] we can now combine Equation (2) and Equation (4) to write

$$\frac{dV}{dt} = KV^{\frac{2}{3}}\Delta\theta \tag{5}$$

where the constant of proportionality K (equals $k\alpha$) depends on the physical properties of ice, the overall shape of the block of ice (cube versus sphere etc), and the units of measurement (e.g., seconds, metres etc).

We can now readily integrate Eq.(5)

$$\int_{V_0(t)}^{V_f(t)} \frac{dV}{V^{\frac{2}{3}}} = K\Delta\theta \int_0^{\tau} dt \tag{6}$$

$$3(V_f^{\frac{1}{3}} - V_0^{\frac{1}{3}}) = (K\Delta\theta)\tau \tag{7}$$

[2] The shape preserving assumption held up well for a few days after the meltdown began but towards the end of the melt the shape was much more like a pyramid. I recommend that you view the security camera images on the Tim-Tam web site to see this effect.

where V_0 is the original volume and V_f the final volume (at time τ). This gives the expression for the melting time as

$$\tau = \frac{3(V_f^{\frac{1}{3}} - V_0^{\frac{1}{3}})}{K\Delta\theta} \tag{8}$$

We now consider the case of a rectangular prism with dimensions $a \times a \times 2a$; hence $V^{\frac{1}{3}} = 2^{\frac{1}{3}}a$ and

$$\tau = \frac{2^{\frac{1}{3}}3(a_f - a_0)}{K\Delta\theta}. \tag{9}$$

Since the Tim-Tam packet is centrally located we will *assume* that when $a_f = a_0/2$ the packet will slide off the top of the ice to the floor. This yields the result

$$\tau = \frac{-3a_0}{2^{\frac{2}{3}}K\Delta\theta} \tag{10}$$

So if all of our assumptions are reasonable it is now just a matter of finding K and then we can have a crack at the $1,000,000. The only way to find K is to proceed empirically. Suppose we take a number of blocks of ice with dimensions approximately $a \times a \times 2a$ (over a range of sizes) and we find the time for them to melt at room temperature θ (over a range of temperatures). A plot of the melting time τ versus $a/\Delta\theta$ should yield a scatter of points about a straight line with slope $-2^{1/3}3/K$ from which we could deduce the constant K. If the points are not scattered about a straight line then it is time to abandon or revise our model. This is an interesting exercise for the reader. You will need to be patient though. The average lunch box sized block of ice takes about a day to melt.

One data point that we do have is from the Tim-Tam competition itself. The packet hit the floor some 9 days, 18 hours, 55 minutes and 58 seconds after the meltdown commenced. That's about 845,758 seconds. Using the approximate value $a \approx 1.8 \times 10^3$ mm we deduce

$$K = -.000201\text{mm} \quad \text{s}^{-1}\text{C}^{-1} \tag{11}$$

How does this K compare with your empirical estimate? With a limit of 200 entries you would like your time estimate to be accurate to within 200 seconds. Given that the known melting time was 845,758 seconds that only allows for a .02% error. Given all the other uncertainties the odds of winning the Tim-Tam Competition were starting to look like the odds of winning Lotto. But I couldn't resist having a crack at it. My estimate after a few poorly controlled experiments over the kitchen sink missed out by about a day. That's only a 10% error. Not bad for a simple mathematical model but no where near the $1,000,000.

Below are two pictures showing security camera photos of the ice inside the fridge after various melt times:

(a) 2 days, 12 hours

(b) 9 days, 17 hours.