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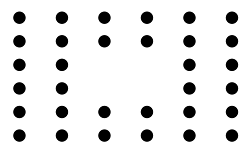
SOLUTIONS TO PROBLEMS 1105-1110

Q1105 A *hollow square* is an arrangement of dots in a square with a central square left blank. For example here are thirty two dots arranged in a hollow square.



In how many different ways can 960 dots be formed into a hollow square.

ANS. Suppose the outer square is $a \times a$ and the inner square is $b \times b$. Then we are seeking positive integer solutions to

$$
\begin{aligned}
960 + b^2 &= a^2 \\
a^2 - b^2 &= 960 \\
(a+b)(a-b) &= 960
\end{aligned}
$$

Now $960 = 2^6 \times 3 \times 5$ has $(6+1)(1+1)(1+1) = 28$ divisors. (How many divisors (factors) does $n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$ have?)

Now at least one of $a+b$, $a-b$ is even, so they are both even. Also $a - b < a + b$, so $a - b < 31$. Hence there are exactly ten solutions

$a-b$	$a+b$	a	b
2	480	241	239
6	160	83	77
10	96	53	43
30	32	31	1
4	240	122	118
12	80	46	34
20	48	34	14
8	120	64	56
24	40	32	8
16	60	38	22

Q1106 How many different ways are there of making six prime numbers which together use each of the nine digits $1, 2, 3, \ldots, 9$ exactly once?

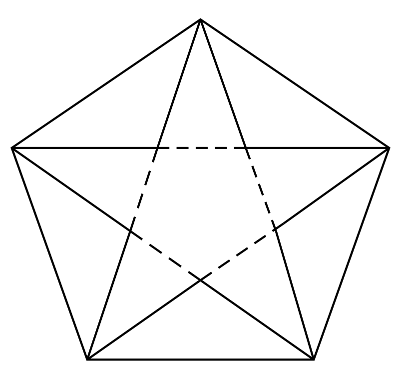
ANS. There are exactly 4 single digit primes, namely $2, 3, 5$ and 7. Also if we only used 2 single digit primes we would need at least ten digits so each solution must include either 3 or 4 single digit primes. Next we note that every prime greater than 10 has $1, 3, 7$ or 9 as its last digit. This means that there are 4 cases

CASE 1	2, 3, 5, 7, x1, xx9
CASE 2	2, 3, 5, 7, x9, xx1
CASE 3	2, 3, 5, x1, x7, x9
CASE 4	2, 5, 7, x1, x3, x9

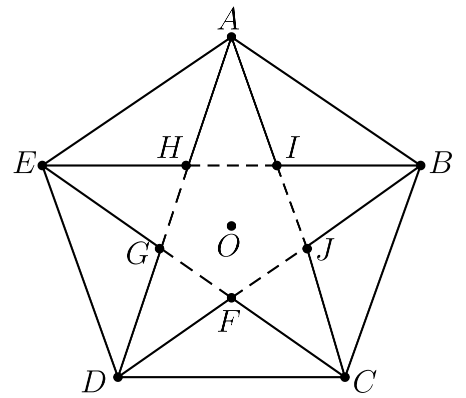
where the crosses are $4, 6$ and 8 in some order. Note that $41, 61$; $43, 83$; $47, 67$ and 89 are all primes.
In CASE 1 we only have to consider $689 = 13 \times 53$, $869 = 11 \times 79$, $489 = 3 \times 163$ and $849 = 3 \times 283$, so there are no solutions in this case.
In CASE 2, both 461 and 641 are prime.
In both CASE 3 and CASE 4, 89 must occur and we find 3 more solutions, giving 5 in all:

$$2,\ 3,\ 5,\ 7,\ 89,\ 461$$
$$2,\ 3,\ 5,\ 7,\ 89,\ 641$$
$$2,\ 3,\ 5,\ 41,\ 67,\ 89$$
$$2,\ 3,\ 5,\ 47,\ 61,\ 89$$
$$2,\ 5,\ 7,\ 43,\ 61,\ 89$$

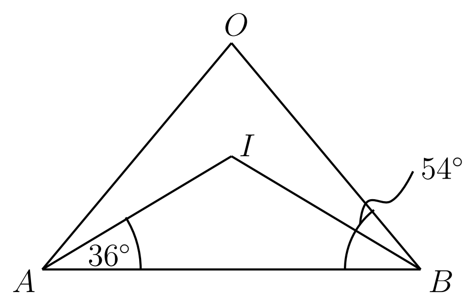
Q1107 Is the large pentagon more than twice, or less than twice, the area of the star inside it?



ANS. Consider the following labellings where O is the centre of the star and both pentagons.



The sum of the internal angles in an n-gon is $(n-2) \times 180°$.
Hence each internal angle in a regular pentagon is $(5-2)180/5 = 108°$.
Thus $\angle AIB = \angle HIJ = 108°$ and $\angle ABI = \angle BAI = 36°$.
On the other hand $\angle AOB = 360/5 = 72°$, so $\angle BAO = (180-72)/2 = 54°$.



We can assume $AB = 2$ and so area $\triangle ABI = \frac{1}{2} \times 2 \times \tan 36° = \tan 36°$ and area $\triangle ABO = \frac{1}{2} \times 2 \times \tan 54° = \tan 54°$. Since the area of the star is 5 times the area of the region $AOBI$ and the area of the large pentagon is 5 times the area of the triangle AOB, the area of the large pentagon is more than twice the area of the star if $\tan 54° > 2(\tan 54° - \tan 36°)$ or $\tan 54° < 2 \tan 36°$.
Now my calculator tells me $\tan 54° = 1.3764$ and $\tan 36° = 0.7265$.
So the pentagon is larger than twice the star.
Note: Those readers who know some trigonometry will appreciate the following explicit calculation of the tangents. It is known that

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

hence

$$\begin{aligned}\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\ &= \frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B} \\ &= \frac{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}{1-\frac{\sin A\sin B}{\cos A\cos B}}\end{aligned}$$

(by dividing both numerator and denominator by $\cos A\cos B$)

$$= \frac{\tan A+\tan B}{1-\tan A\tan B}$$

In particular,

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

If we let $t=\tan 36°$, then $\tan 54° = \cot 36° = 1/\tan 36° = 1/t$.
Next $\tan 72° = \frac{2t}{1-t^2}$, so $\tan 18° = \cot 72° = \frac{1-t^2}{2t}$.
However $36° = 2\times 18°$, so

$$t = \frac{\frac{2(1-t^2)}{2t}}{1-\frac{(1-t^2)^2}{(2t)^2}} = \frac{4t(1-t^2)}{4t^2-(1-t^2)^2}.$$

Hence

$$\begin{aligned}4t^2-1+2t^2-t^4 &= 4-4t^2 \\ t^4-10t^2+5 &= 0 \\ (t^2-5)^2 &= 20 \\ t^2 &= 5\pm 2\sqrt{5}.\end{aligned}$$

But $\tan 36° < \tan 45° = 1$. Hence $t^2 = 5-2\sqrt{5}$, so $t=\sqrt{5-2\sqrt{5}}$.
Finally $\frac{1}{t} < 2t \Leftrightarrow t^2 > \frac{1}{2} \Leftrightarrow 5-2\sqrt{5} > \frac{1}{2} \Leftrightarrow 9/2 > 2\sqrt{5} \Leftrightarrow 81/4 > 20$, as required.

SECOND ANSWER
Returning to the original diagram,
$\angle FBC = \angle IBA = 36°$, so $\angle FBI = 108° - 72° = 36° = \angle ABI$.
Also $\angle IJF = 108°$ and $\triangle IJF$ is isoceles, so $\angle BFI = \angle FIJ = 36° = \angle BAI$.
Hence $\triangle FIB \equiv \triangle AIB$.
It is also easy to see that O is closer to IJ than F. Hence area $\triangle OIJ <$ area $\triangle FIJ$ and hence area region $OJBI <$ area $\triangle FIB =$ area $\triangle AIB$.
However $OJBI$ is $1/5$ of the star, hence result.

Q1108 It is a curious fact that $\sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}}$.
Is this isolated or are there other such expressions?
Find all solutions to $\sqrt{m+x} = m\sqrt{x}$ where m is a positive integer and x is real.
ANS. As we shall see this is actually a problem in number theory.
Suppose

$$\begin{aligned}\sqrt{m+x} &= m\sqrt{x}\\ m+x &= m^2x\\ x &= \frac{m}{m^2-1}\end{aligned}$$

so x is actually a rational number.
Conversely if $x = \frac{m}{m^2-1}$ then $\sqrt{m+x} = m\sqrt{x}$ so we have an infinite family of solutions, one for each integer $m \geq 2$. For example $\sqrt{7+\frac{7}{48}} = 7\sqrt{\frac{7}{48}}$.

Q1109 Determine the smallest value of $x^2+5y^2+8z^2$, where $x,\ y$ and z are real numbers subject to the condition $yz+zx+xy=-1$. Does $x^2+5y^2+8z^2$ have a *greatest* value subject to the same condition? Justify both answers.

ANS. Intuitively one would expect this expression to have a smallest value since x, y and z cannot all approach 0 when $yz+zx+xy=-1$. However, one would expect it to grow without bound in some part of the surface $yz+zx+xy=-1$.

In fact, the second part of the problem is relatively straight forward. For if we set $z=0$ we obtain a rectangular hyperbola $xy=-1$. So, for any value of α with $\alpha > 0$, the points $(\alpha, -1/\alpha, 0)$ all lie on the surface and $x^2+5y^2+8z^2$ is greater than α^2. So the function has no greatest value.

The first part is much harder. To motivate the solution consider

$$\begin{aligned}x^2+4xy+9y^2 &= (x+2y)^2+5y^2\\ x^2+4xy+y^2 &= (x+2y)^2-3y^2.\end{aligned}$$

The first quadratic is expressed as the *sum* of two squares and hence is never negative, whereas the second quadratic is expressed as the *difference* of two squares and is sometimes positive and sometimes negative. (The first is called "positive definite" and the second is "indefinite".)
To tackle $x^2+5y^2+8z^2$ subject to $yz+zx+xy=-1$ one tries to express

$$x^2+5y^2+8z^2 \equiv (\alpha x+\beta y+\gamma z)^2+(\delta x+\varepsilon y+\eta z)^2+\zeta(yz+zx+xy).$$

If we can calculate these 7 constants and obtain a negative value of ζ then a solution is close. Clearly there are too many constants to calculate and a little thought suggests trying

$$
\begin{aligned}
x^2+5y^2+8z^2 &\equiv (x+by+bz)^2+(cy+dz)^2+e(yz+zx+xy) \\
&\equiv x^2+b^2y^2+b^2z^2+2bxy+2bxz+2b^2yz+c^2y^2+2cdyz+d^2z^2 \\
&\quad +eyz+ezx+exy \\
&\equiv x^2+(b^2+c^2)y^2+(b^2+d^2)z^2+(2b+e)xy+(2b+e)xz+(2b^2+2cd+e)yz
\end{aligned}
$$

as an identity in x, y and z. Now equate coefficients obtaining

$$
\begin{aligned}
b^2+c^2 &= 5 && (1)\\
b^2+d^2 &= 8 && (2)\\
2b+e &= 0 && (3)\\
2b^2+2cd+e &= 0 && (4)
\end{aligned}
$$

Eliminating e from (3) and (4) yields

$$
\begin{aligned}
2b^2+2cd-2b &= 0 \\
cd &= b-b^2 \\
c^2d^2 &= b^2-2b^3+b^4 && (5)
\end{aligned}
$$

Next we substitute (1) and (2) into (5) eliminating c^2 and d^2.

$$(5-b^2)(8-b^2)=b^2-2b^3+b^4$$

$$
\begin{aligned}
2b^3-14b^2+40 &= 0 \\
b^3-7b^2+20 &= 0 \\
(b-2)(b^2-5b-10) &= 0
\end{aligned}
$$

or

$$b=2, \quad \frac{5\pm\sqrt{25+40}}{2}.$$

We consider $b=2$ first and (1), (2), (3) yield $c=\pm 1$, $d=\pm 2$, $e=-4$. Now (4) shows that c and d have opposite signs, so returning to the identity we have

$$
\begin{aligned}
x^2+5y^2+8z^2 &\equiv (x+2y+2z)^2+(y-2z)^2-4(yz+zx+xy) \\
&= (x+2y+2z)^2+(y-2z)^2+4
\end{aligned}
$$

We must finish the problem by showing the apparent minimum value 4 can be achieved. So assume $z=t$ and $y-2z=x+2y+2z=0$ then $y=2t$ and $x=-6t$. Substituting these values in the surface,

$$2t^2-6t^2-12t^2=yz+zx+xy=-1 \qquad \text{or} \qquad t=\pm\frac{1}{4}.$$

In conclusion we have shown that the minimum is 4 and it occurs at $\pm(\frac{3}{2}, -\frac{1}{2}, -\frac{1}{4})$.

Note:
The reader should consider the meaning of the other roots $\frac{5\pm\sqrt{65}}{2}$ of the cubic.

Q1110 Let f be a function mapping positive integers into positive integers. Suppose that $f(n+1) > f(n)$ and $f(f(n)) = 3n$ for all positive integers n. Determine $f(2001)$.

ANS. The standard approach to solving a "functional equation" problem is to begin by determining possible values of $f(0)$, $f(1)$, etc. or by showing that f has other properties. Once enough is known about f it should be possible to conjecture some more general properties of f and prove them, often by using induction.

STEP 1 $f(1) > 1$. For suppose $f(1) = 1$. Then $3 = f(f(1)) = f(1) = 1$, a contradiction.

STEP 2 $f(n) > n$. For suppose $f(k) \le k$ for some k. Then $f(k-1) < f(k) \le k$, so $f(k-1) \le k-1$. Repeating this argument $k-1$ times we obtain $f(1) \le 1$ in contradiction to step 1. Hence $f(n) > n$ for all n.

STEP 3 $f(1) = 2$. For suppose $f(1) = n \ge 3$. Then $3 = f(f(1)) = f(n) > n \ge 3$, a contradiction. Hence $f(1) = 2$.

It is now possible to calculate a number of values.

$$\begin{array}{lllllll} f(2) &=& f(f(1)) &=& 3, & f(3) &=& f(f(2)) &=& 6, \\ f(6) &=& f(f(3)) &=& 9, & f(9) &=& f(f(6)) &=& 18, \\ f(18) &=& f(f(9)) &=& 27, & f(27) &=& f(f(18)) &=& 54 \end{array}$$

and a pattern is clear. We conjecture $f(3^n) = 2 \times 3^n$ and $f(2 \times 3^n) = 3^{n+1}$.

Next we see that $6 = f(3) < f(4) < f(5) < f(6) = 9$, hence $f(4) = 7, f(5) = 8$ and $f(7) = f(f(4)) = 12$, $f(8) = f(f(5)) = 15$.
At this stage it is reasonably clear that there is only one function f which satisfies both conditions and perhaps a table of values will help.

n	$f(n)$	n	$f(n)$	n	$f(n)$	n	$f(n)$
1	2	8	15	15		22	
2	3	9	18	16		23	
3	6	10		17		24	
4	7	11		18	27	25	
5	8	12		19		26	
6	9	13		20		27	54
7	12	14		21		28	

There are eight blanks between 9 and 18 and eight numbers between 18 and 27. Hence $f(10) = 19$, $f(11) = 20$, $\ldots$, $f(17) = 26$. Next

$$f(19) = f(f(10)) = 30, \quad f(20) = f(f(11)) = 33, \quad \ldots, \quad f(26) = f(f(17)) = 51,$$

$$f(30) = f(f(19)) = 57, \quad f(33) = f(f(20)) = 60, \quad \ldots, \quad f(51) = f(f(26)) = 78, \qquad \text{etc.}$$

So we get the following table:

n	$f(n)$	n	$f(n)$	n	$f(n)$	n	$f(n)$
1	2	15	24	29		43	
2	3	16	25	30	57	44	
3	6	17	26	31		45	72
4	7	18	27	32		46	
5	8	19	30	33	60	47	
6	9	20	33	34		48	75
7	12	21	36	35		49	
8	15	22	39	36	63	50	
9	18	23	42	37		51	78
10	19	24	45	38		52	
11	20	25	48	39	66	53	
12	21	26	51	40		54	81
13	22	27	54	41		55	
14	23	28		42	69	56	

The rest of the above table can now be filled in. So we conjecture

$$\begin{aligned} f(3^n) &= 2 \times 3^n \\ f(2 \times 3^n) &= 3^{n+1} \\ f(3^n + k) &= 2 \times 3^n + k \qquad 1 \le k < 3^n \\ f(2 \times 3^n + k) &= 3^{n+1} + 3k \qquad 1 \le k < 3^n \end{aligned}$$

The proofs of these are easy by induction and are left to the reader. Finally $3^6 = 729$, $2.3^6 = 1458$ and $3^7 = 2187$, so $2001 = 2.3^6 + 543$ and

$$f(2001) = f(2 \times 3^6 + 543) = 3^7 + 3 \times 543 = 3816.$$