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Solutions to Problems 1151-1160

Q1151 Let $p(x) = (x^{2003} + x^{2002} - 1)^{2004}$. Find the sum of the coefficients of all odd degree terms in the expansion of the trinomial $p(x)$.

ANS. The expansion of $p(x)$ takes the form

$$p(x) = a_0 + a_1^x + a_2x^2 + \cdots + a_nx^n$$

where $n = 2003 \times 2004$ is even. So the sum of the coefficients of all odd degree terms is

$$S = a_1 + a_3 + a_5 + \cdots + a_{n-1}.$$

Since

$$\begin{aligned} p(1) &= a_0 + a_1 + \cdots + a_{n-1} + a_n \\ p(-1) &= a_0 - a_1 + \cdots - a_{n-1} + a_n \end{aligned}$$

we have

$$\begin{aligned} S &= \frac{p(1) - p(-1)}{2} \\ &= \frac{(1^{2003} + 1^{2002} - 1)^{2004} - (-1^{2003} + 1^{2002} - 1)^{2004}}{2} \\ &= 0. \end{aligned}$$

Q1152 Find the coefficients of x^{n-1} and x^{n-2} in the expansion of

$$p(x) = (x + \frac{1}{2})(x + \frac{1}{2^2})(x + \frac{1}{2^3}) \cdots (x + \frac{1}{2^n}).$$

ANS. The coefficient a_{n-1} of x^{n-1} is

$$\begin{aligned} a_{n-1} &= \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \\ &= \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right) \\ &= \frac{1}{2}\left(\frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}}\right) \\ &= 1 - \frac{1}{2^n}. \end{aligned}$$

The coefficient a_{n-2} of x^{n-2} is

$$\begin{aligned}a_{n-2} = & \frac{1}{2}\cdot\frac{1}{2^2}+\frac{1}{2}\cdot\frac{1}{2^3}+\cdots+\frac{1}{2}\cdot\frac{1}{2^n}\\ & +\frac{1}{2^2}\cdot\frac{1}{2^3}+\frac{1}{2^2}\cdot\frac{1}{2^4}+\cdots+\frac{1}{2^2}\cdot\frac{1}{2^n}\\ & +\cdots\\ & +\frac{1}{2^{n-1}}\cdot\frac{1}{2^n}.\end{aligned}$$

So

$$\begin{aligned}2a_{n-2} &= \left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^n}\right)^2-\left[\left(\frac{1}{2}\right)^2+\left(\frac{1}{2^2}\right)^2+\cdots+\left(\frac{1}{2^n}\right)^2\right]\\ &= a_{n-1}^2-\left(\frac{1}{4}+\frac{1}{4^2}+\cdots+\frac{1}{4^n}\right)\\ &= \left(1-\frac{1}{2^n}\right)^2-\frac{1}{4}\left(1+\frac{1}{4}+\cdots+\frac{1}{4^{n-1}}\right)\\ &= \left(1-\frac{1}{2^n}\right)^2-\frac{1}{4}\cdot\frac{1-\left(\frac{1}{4}\right)^n}{1-\frac{1}{4}}\\ &= \left(1-\frac{1}{2^n}\right)^2-\frac{1}{3}\left(1-\frac{1}{4^n}\right)\end{aligned}$$

therefore

$$\begin{aligned}a_{n-2} &= \frac{1}{2}\left[\left(1-\frac{1}{2^n}\right)^2-\frac{1}{3}\left(1-\frac{1}{4^n}\right)\right]\\ &= \frac{1}{2}\left(1-\frac{1}{2^{n-1}}+\frac{1}{2^{2n}}-\frac{1}{3}+\frac{1}{3.4^n}\right)\\ &= \frac{1}{3}-\frac{1}{2^n}+\frac{2}{3\times 4^n}.\end{aligned}$$

Q1153 Find all values of a (real number) such that the system

$$x^3-ay^3 = \frac{1}{2}(1+a)^2 \qquad (1)$$

$$x^3+ax^2y+xy^2 = 1 \qquad (2)$$

has a solution, and that **all** solutions satisfy

$$x+y=0. \qquad (3)$$

ANS. Assume that (x, y) is a solution to (1) and (2) and that x and y satisfy (3), then $y = -x$. Substituting this into (1) and (2) gives

$$(1+a)x^3 = \frac{1}{2}(1+a)^2 \quad (4)$$
$$(2-a)x^3 = 1. \quad (5)$$

It follows from (5) that $a \neq 2,\ x \neq 0$ and

$$x^3 = \frac{1}{2-a} \quad (6)$$

(4) and (6) imply

$$\frac{1+a}{2-a} = \frac{1}{2}(1+a)^2$$

or, equivalently,

$$(1+a)\frac{a(a-1)}{2(2-a)} = 0.$$

There are 3 values of $a : a = 0, a = -1, a = 1$.

Case 1: $a = 0$
Systems (1) and (2) become

$$\begin{cases} x^3 = \frac{1}{2} \\ x^3 + xy^2 = 1 \end{cases}$$

and this system has as solutions: $\left(\frac{1}{\sqrt[3]{2}}\right)$, and $\left(\frac{1}{\sqrt[3]{2}}, -\frac{1}{\sqrt[3]{2}}\right)$.
Not all solutions satisfy (3).

Case 2: $a = -1$
System (1) and (2) become

$$x^3 + y^3 = 0$$
$$x^3 - x^2y + xy^2 = 1$$

this system has a unique solution $\left(\frac{1}{\sqrt[3]{2}}, -\frac{1}{\sqrt[3]{2}}\right)$ and this solution satisfies (3).

Case 3: $a = 1$
Systems (1) and (2) become

$$x^3 - y^3 = 2 \quad (7)$$
$$x^3 + x^2 2y + xy^2 = 1. \quad (8)$$

Let $y = tx$ then (7) and (8) become

$$x^3(1-t^3) = 2 \quad (9)$$
$$x^3(1+t+t^2) = 1.$$

This implies

$$\frac{1-t^3}{1+t+t^2} = 2$$

or equivalently

$$(1+t)(1+t+t^2) = 0.$$

This equation has a unique solution $t = -1$. Substituting into (9) we obtain $x = 1$ and thus $y = -1$.
This unique solution $(1, -1)$ of (1) and (2) satisfies (3).

Answer: $a = 1$ and $a = -1$.

Q1154 Let $a, b,$ and c be three positive numbers. Prove that

$$\frac{1}{a^2+bc} + \frac{1}{b^2+ca} + \frac{1}{c^2+ab} \leq \frac{a+b+c}{2abc} \tag{1}$$

and

$$\frac{1}{a^3+b^3+abc} + \frac{1}{b^3+c^3+abc} + \frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}. \tag{2}$$

ANS. Proof of (1):
By Cauchy's inequality we have

$$\begin{aligned} a^2+bc &\geq 2a\sqrt{bc} \\ b^2+ca &\geq 2b\sqrt{ca} \\ c^2+ab &\geq 2c\sqrt{ab} \end{aligned}$$

therefore

$$\frac{1}{a^2+bc} + \frac{1}{b^2+ca} + \frac{1}{c^2+ab} \leq \frac{1}{2a\sqrt{bc}} + \frac{1}{b^2+ca} + \frac{1}{c^2+ab}$$

$$\begin{aligned} &\leq \frac{\sqrt{bc}+\sqrt{ca}+\sqrt{ab}}{2abc} \\ &\leq \frac{\frac{b+c}{2}+\frac{c+1}{2}+\frac{a+b}{2}}{2abc} \qquad \text{(Cauchy's inequality)} \\ &= \frac{a+b+c}{2abc}. \end{aligned}$$

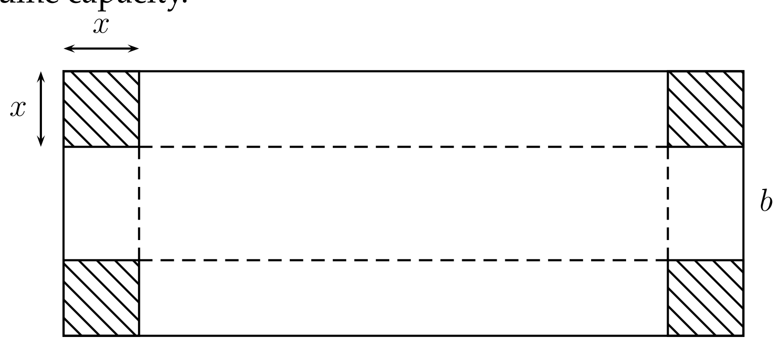
Proof of (2):
For all $x, y > 0$ there holds

$$\begin{aligned} x^3+y^3 &= (x+y)(x^2+y^2-xy) \\ &\geq (x+y)(2xy-xy) \\ &= xy(x+y) \end{aligned}$$

therefore

$$\begin{aligned}
&\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\\
&\qquad\le \frac{1}{ab(a+b)+abc}+\frac{1}{bc(b+c)+abc}+\frac{1}{ca(c+a)+abc}\\
&\qquad= \frac{1}{ab(a+b+c)}+\frac{1}{bc(a+b+c)}+\frac{1}{ca(a+b+c)}\\
&\qquad= \frac{c}{abc(a+b+c)}+\frac{a}{abc(a+b+c)}+\frac{b}{abc(a+b+c)}\\
&\qquad= \frac{1}{abc}.
\end{aligned}$$

Q1155 Four equal squares of side x cm are cut from the corners of a sheet of metal of size a cm and b cm, as in the figure. We then fold the sheet along the dotted lines to make a container (without a lid). Find the value of x so that the container has a maximum volume capacity.



ANS. Without loss of generality, we assume that $b < a$. Then $0 < x < \frac{b}{2}$, and the volume capacity is

$$V = x(a-2x)(b-2x).$$

By differentiating with respect to x we obtain

$$V' = 12x^2 - 4(a+b)x + ab.$$

Critical points:

$$\begin{aligned}
x_1 &= \frac{4(a+b)-\sqrt{16(a+b)^2-48ab}}{24} = \frac{a+b-\sqrt{a^2+b^2-ab}}{6}\\
x_2 &= \frac{4(a+b)+\sqrt{16(a+b)^2-48ab}}{24} = \frac{a+b+\sqrt{a^2+b^2-ab}}{6}.
\end{aligned}$$

(Note that $a^2+b^2-ab > 0$.)
We need to check which of the above solutions satisfy

$$0 < x < \frac{b}{2}.$$

We have

$$
\begin{aligned}
x_2 - \frac{6}{2} &= \frac{a - 2b + \sqrt{a^2 + b^2 - ab}}{6} \\
&> \frac{a - 2b + \sqrt{a^2 + b^2 - a^2}}{6} \qquad \text{(because } b < a\text{)} \\
&= \frac{a - b}{6} \\
&> 0, \\
\text{i.e.} \quad x_2 &> \frac{6}{2}.
\end{aligned}
$$

We will now prove that $x_1 < \frac{b}{2}$, or equivalently

$$a - 2b < \sqrt{a^2 + b^2 - ab}. \tag{1}$$

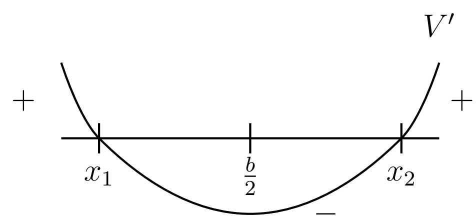
If $a - 2b \leq 0$ then the above inequality is true. If $a - 2b > 0$ then by squaring both sides we have

$$a^2 - 4ab + 4b^2 < a^2 + b^2 - ab$$

or

$$b^2 < ab.$$

The final inequality is true. So (1a) is true. The graph of V' is



So in $[0, \frac{6}{2}]$ V behaves like

x	0		x_1		$\frac{b}{2}$
V'		$+$	0	$-$	
V	↗		$V_{\max}$		↘

So V attains its maximum value when $x = x_1$.

Q1156 Two friends Jack and Jane have to take a train at the same station to get to school. To ensure not to be late, they have to be at the station some time between 8am and 8:30am. They want to travel together on the train, but don't want to wait too long for each other. So they come to this agreement. Each one arriving at the station will wait at most 5 minutes for the other. What is the probability that they travel together?

ANS. Let x (respectively y) be the time (in minutes) after 8am that Jack (respectively Jane) arrives at the station. Then

$$0 \le x,\ y \le 30.$$

They will travel on the same train if (and only if)

$$|x - y| \le 5$$

or equivalently

$$x - 5 \le y \le x + 5.$$

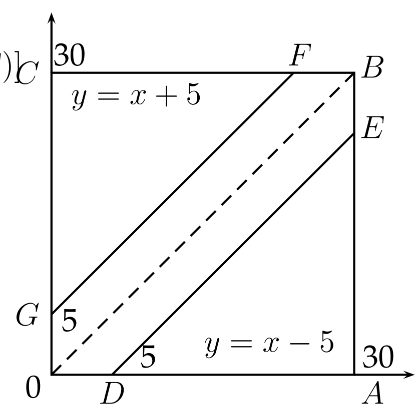
The problem is now equivalent to the following problem. Let (x, y) be a point randomly chosen in the square $OABC$ (see figure). Find the probability that (x, y) is in the polygon $ODEBFG$.

The area of the polygon $ODEBFG$ is

$$\begin{aligned} S_1 &= 2 \times [\text{area } (OAB) - \text{ area } (DAE)] \\ &= 2\left(\tfrac{1}{2} \times 30^2 - \tfrac{1}{2}25^2\right) \\ &= 30^2 - 25^2 \\ &= 275. \end{aligned}$$

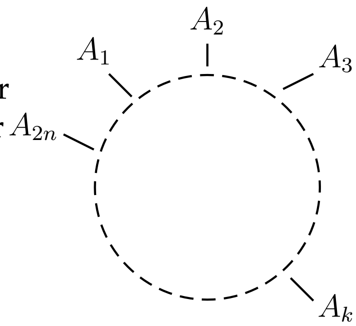
The area of the square $OABC$ is:
$S_2 = 900$. So the probability is

$$p = \frac{275}{900} \simeq 30.6\%.$$



Q1157 Let $A_1, A_2, \cdots, A_{2n}$ $(n \ge 2)$ be $2n$ points (in that order) on a circle such that $A_1A_2 \cdots A_{2n}$ is a regular polygon.

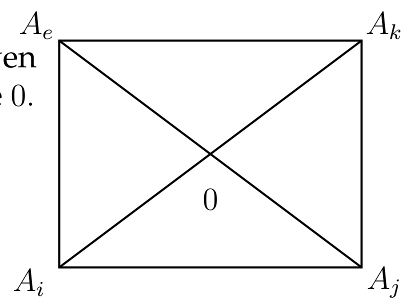
Any 3 points form a triangle, while some set of 4 points form a rectangle. Given that the number of triangles is 20 times the number of rectangles, find n.



ANS. The number of triangles is C_{2n}^3.

It is rather more complicated to count the number of rectangles. Four points A_i, A_j, A_k, and A_ℓ form a rectangle $A_iA_jA_kA_\ell$ if and only if the diagonals A_iA_k and A_jA_ℓ pass through the centre 0 of the circle.

So to count the number of rectangles we count all the diagonals, of the given polygon that pass through the centre 0. Since the polygon is regular and has $2n$ sides, we have $\measuredangle A_i 0 A_{i+1} = \frac{360°}{2n} = \frac{180°}{n}$, $i = 1, 2, 3,$ therefore,



$$\begin{aligned}\measuredangle A_1 0 A_{1+n} &= \measuredangle A_1 0 A_2 + \measuredangle A_2 0 A_3 + \cdots + \measuredangle A_n 0 A_{1+n} \\ &= \underbrace{\frac{180°}{n} + \cdots + \frac{180°}{n}}_{n \text{ times}} \\ &= 180°,\end{aligned}$$

i.e. A_1A_{1+n} is a diagonal passing through the centre. Similarly, $A_2A_{2+n}, A_3A_{3+n}, \cdots, A_nA_{2n}$ are diagonals passing through 0, i.e. there are n such diagonals. Any two of these diagonals form a rectangle, so the number of rectangles is C_n^2. By assumption

$$C_{2n}^3 = 20 \times C_n^2$$

or

$$\frac{(2n)!}{3!(2n-3)!} = 20 \times \frac{n!}{2!(n-2)!}.$$

Simplifying the fractions and rearranging the terms we obtain

$$n^2 - 9n + 8 = 0$$

i.e. $n = 1$ or $n = 8$. Since $n \geq 2$ we have $n = 8$.

Q1158 Let $S = \frac{1}{1.2.3} + \frac{1}{4.5.6} + \cdots + \frac{1}{2002.2003.2004}$.

Prove that

$$S = \frac{2005 \times A}{1.2.3 \cdots 2002.2003.2004}$$

for some integer A.

ANS. It is easy to see that each term in the sum defining S has the form

$$x_k = \frac{1}{(3k-2)(3k-1)(3k)}, \; k = 1, \cdots, 668.$$

We write S as

$$S = x_1 + x_2 + \cdots + x_{667} + x_{668}$$

or as

$$S = x_{668} + x_{667} + \cdots + x_2 + x_1.$$

We then deduce

$$2S = (x_1 + x_{668}) + (x_2 + x_{667}) + \cdots + (x_{667} + x_2) + (x_{668} + x_1).$$

Each pair in the parentheses has the form

$$y_k = x_k + x_{669-k}, \; k = 1, 2, \cdots, 668.$$

By symmetry,

$$2S = 2(y_1 + \cdots + y_{334})$$

or

$$S = y_1 + \cdots + y_{334}.$$

If we can prove that

$$y_k = \frac{2005 \, A_k}{(3k-2)(3k-1)(3k)(2005-3k)(2006-3k)(2007-3k)} \qquad (2)$$

where A_k is an integer, then

$$S = 2005\Big[\frac{A_1}{(1.2.3)(2002.2003.2004)} + \frac{A_2}{(4.5.6)(1999.2000.2001)} + \frac{A_{334}}{(1000.1001.1002)(1003.1004.1005)}\Big].$$

By writing the sum in the square brackets as a single fraction with common denominator

$$(1.2.3)(4.5.6) \cdots (2002.2003.2004)$$

we obtain

$$S = \frac{2005 \, A}{1.2.3 \cdots 2003.2004},$$

for some integer A.

We now prove (2). From the definitions of x_k and y_k we have

$$\begin{aligned} y_k &= \frac{1}{(3k-2)(3k-1)(3k)} + \frac{1}{(2005-3k)(2006-3k)(2007-3k)} \\ &= \frac{(2005-3k)(2006-3k)(2007-3k) + (3k-2)(3k-1)(3k)}{(3k-2)(3k-1)(3k)(2005-3k)(2006-3k)(2007-3k)}. \end{aligned}$$

Let $C_k = (2005-3k)(2006-3k)(2007-3k) + (3k-2)(3k-1)(3k)$.

It suffices to prove

$$C_k = 2005\ A_k$$

for some integer A_k. We have

$$\begin{aligned} C_k &= (2006-3k)[(2006-3k)-1][(2006-3k)+1] \\ &\quad + (3k-1)[(3k-1)-1][(3k-1)+1] \\ &= (2006-3k)[(2006-3k)^2-1] + (3k-1)[(3k-1)^2-1] \\ &= (2006-3k)^3 + (3k-1)^3 - [(2006-3k)+(3k-1)]. \end{aligned}$$

By using the formula $a^3+b^3 = (a+b)(a^2-ab+b^2)$ we obtain

$$C_k = 2005\ A_k$$

with

$$A_k = (2006-3k)^2 - (2006-3k)(3k-1) + (3k-1)^2 - 1.$$

Q1159 Prove that there exists a triangle ABC such that its angles are solutions of the equation

$$(56 - 65\sin x)(80 - 64\sin x - 65\cos^2 x) = 0. \tag{1}$$

ANS. We first find all the solutions of (1). Equation (1) is equivalent to

$$56 - 65\sin x = 0 \tag{2}$$

or

$$80 - 64\sin x - 65\cos^2 x = 0. \tag{3}$$

From (2) we have $\sin x = \dfrac{56}{65}$.

From (3) we have

$$65\sin^2 x - 64\sin x + 15 = 0$$

so

$$\sin x = \frac{3}{5} \quad \text{or} \quad \sin x = \frac{5}{13}.$$

Since $\frac{3}{5} < \frac{1}{\sqrt{2}}$ and $\frac{5}{13} < \frac{1}{2}$, there exist $B \in (0, \frac{\pi}{4})$ and $C \in (0, \frac{\pi}{6})$ such that $\sin B = \frac{3}{5}$ and $\sin C = \frac{5}{13}$. Then $\cos B = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$ and $\cos C = \sqrt{1 - \left(\frac{5}{13}\right)^2} = \frac{12}{13}$.

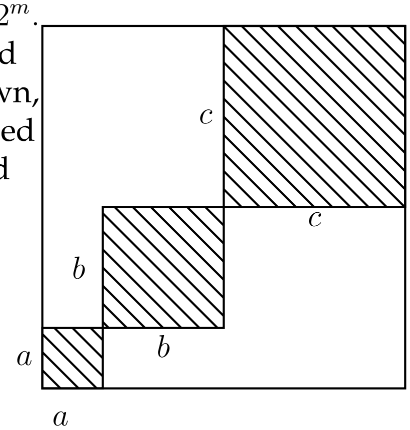
So

$$\begin{aligned}\sin(B+C) &= \sin B\cos C + \sin C\cos B\\ &= \frac{3}{5}\times\frac{12}{13}+\frac{5}{13}\times\frac{4}{5}\\ &= \frac{56}{65}.\end{aligned}$$

Note that $0 < B+C < \dfrac{\pi}{2}$. Let $A = \pi - (B+C)$. Then $\sin A = \frac{56}{65}$. A, B, C can be three angles of a triangle!

Q1160 Jim has at the back of his house a piece of land of size $3^m \times 3^m$. He designs this land as in the figure, where he wants to pave the shaded area and grow lawn on the remainder.

Assume that $0 \le a < b < c \le 2^m$. Since he doesn't want to spend too much time to mow the lawn, he wants to maximise the paved area. Find the value of a, b and c such that the paved area is a maximum. What is this maximal area?



ANS. The paved area is

$$S = a^2 + b^2 + c^2$$

let

$$a = 1+\alpha,\ b = 1+\beta,\ c = 1+\gamma.$$

Since $0 \le a < b < c \le 2$, we have

$$-1 \le \alpha < \beta < \gamma \le 1. \tag{1}$$

Also, because $a+b+c = 3$ we have

$$\alpha+\beta+\gamma = 0. \tag{2}$$

Then

$$\begin{aligned}S &= (1+\alpha)^2 + (1+\beta)^2 + (1+\gamma)^2\\ &= 3 + \alpha^2 + \beta^2 + \gamma^2.\end{aligned}$$

$S_1 = \alpha^2 + \beta^2 + \gamma^2$. Due to (1) we have

$$S_1 \le |\alpha| + |\beta| + |\gamma|.$$

Consider all possible cases:
Case 1: $-1 \leq \alpha < \beta \leq 0 < \gamma \leq 1.$
In this case

$$\begin{aligned} S_1 \leq -\alpha - \beta + \gamma &= -(\alpha + \beta + \gamma) + 2\gamma \\ &= 2\gamma \qquad \text{(due to (2))} \\ &\leq 2. \end{aligned}$$

Case 2: $-1 \leq \alpha \leq 0 < \beta < \gamma \leq 1$
In this case

$$\begin{aligned} S_1 \leq -\alpha + \beta + \gamma &= -2\alpha + (\alpha + \beta + \gamma) \\ &= -2\alpha \qquad \text{(due to (2))} \\ &\leq 2. \end{aligned}$$

There are no other cases.
So

$$\begin{aligned} &S_1 \leq 2 \text{ for all } \alpha, \beta, \gamma \text{ satisfying} \\ &\qquad -1 \leq \alpha < \beta < \gamma \leq 1 \text{ and } \alpha + \beta + \gamma = 0. \end{aligned}$$

In the first case, S_1 attains its maximum value 2 when $\gamma = 1$. Then

$$\begin{cases} \alpha + \beta = -1 \\ \alpha^2 + \beta^2 = 1. \end{cases}$$

Substituting $\alpha = -1 - \beta$ into the second equation we obtain

$$\beta(\beta + 1) = 0.$$

Since $\beta > -1$ we have $\beta = 0$, and therefore $\alpha = -1$. So S_1 attains its maximum value 2 when

$$\alpha = -1, \beta = 0, \gamma = 1.$$

Similarly we have, in the second case, $\alpha = -1, \beta = 0, \gamma = 1$. So the paved area is maximised when

$$a = 0, b = 1, c = 2$$

and the maximal area is $S = 5m^2$.