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Trigonometry: Chords, Arcs and Angles

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Trigonometry, as it is taught in high school using the trigonometric ratios, has an interesting history. Indeed, it is a relatively recent invention, going back roughly to the 1400's, although Arab mathematicians developed essentially the same ideas earlier, but written in a form which we would probably not immediately recognise.

An earlier form of trigonometry, however, can be traced back to the ancient Greeks, notably to the two mathematicians Hipparchus and Ptolemy. This version of trigonometry was based on chords in a circle. Hipparchus of Bithynia was an astronomer who was born in 190BC and died in 120BC. He is considered to be one of the most influential of the early astronomers, and is credited with the founding of trigonometry. His work highlighted the need for a system that provided a unit of measure for arcs and angles. The Babylonians divided the circle into 360 parts; the reasons for this, however, are unclear. They may have chosen 360 since it is divisible by many small integers, or, more likely, because 360 is the number of days in a year rounded to the nearest ten.

Hipparchus' trigonometry was based on the chord subtending a given arc in a circle of fixed radius R.

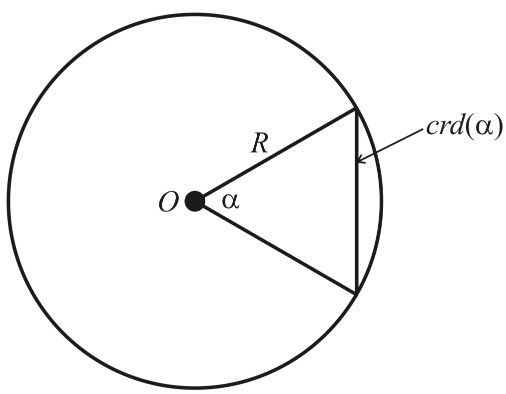

Figure 1: This figure shows the chord $crd(\alpha)$ subtending an angle α in a circle.

The length of the chord is denoted by $crd(\alpha)$.

Hipparchus and later Ptolemy, gave a table listing α and $crd(\alpha)$ for various values of the angle α, based on a specific value of R. Ptolemy used the value $R = 60$, whereas Hipparchus used a more complicated value, as we shall see below.

Using basic circle properties, we can see from the following diagram that $crd(\alpha)$ is related to the sine ratio by the equation

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$\sin\left(\frac{\alpha}{2}\right) = \frac{\frac{1}{2}\,crd(\alpha)}{R}$, so $crd(\alpha) = 2R\sin(\frac{\alpha}{2})$. Hence, in a circle of diameter 1, we have $crd(\alpha) = \sin(\frac{\alpha}{2})$.

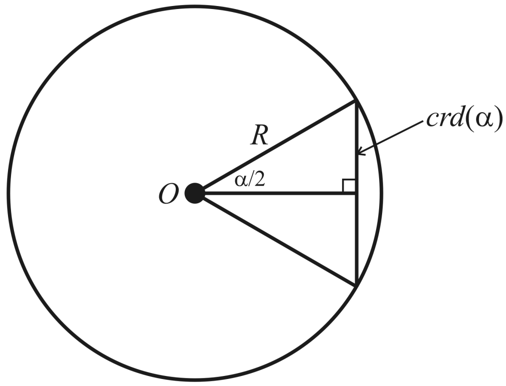

Figure 2: This figure shows the chord $crd(\alpha)$ and the half angle $\alpha/2$.

Also, using the circle property, that the angle at the centre of a circle is twice the angle at the circumference subtended by the same arc, it can be seen that in a circle of diameter 1, the chord which subtends an angle α at the **circumference** has length $\sin\alpha$. This is well-defined, since equal chords subtend equal angles.

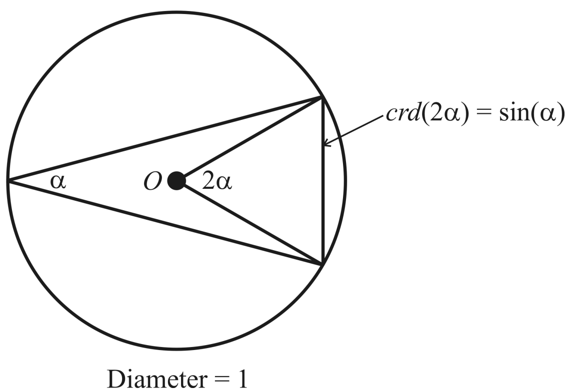

Figure 3: This figure shows the geometry for which chord $crd(\alpha) = \sin(\alpha)$.

Hipparchus knew that $2\pi R$ was equal to the circumference of a circle, and taking $3;8,30$ as the sexagesimal[2] approximation for π, the radius R was calculated:

[2]Sexagesimal represents real numbers in a base 60 system rather than base 10. Time is still measured in a base 60 system today. The notation $3;8,30$ is used to represent $3+\frac{8}{60}+\frac{30}{60^2}$. Ed.

$$
\begin{aligned}
C &= 2\pi R \\
R &= \frac{C}{2\pi} \\
&= \frac{60 \times 360}{2\pi} \\
&= \frac{6,0,0}{6;17} \qquad \text{i.e.} \quad \begin{array}{l}(6,0,0 = 6 \times 60^2 + 0 \times 60 + 0 \times 1) \\ (6;17 = 2 \times 3;8,30 = 2 \times (3 + \frac{8}{60} + \frac{30}{3600})\end{array} \\
&= 57,18 \\
&= 3438'
\end{aligned}
$$

With this radius, the measure of an angle is equal to its radius measure. The measure of an angle is defined as the length cut off on the circumference divided by the radius. In calculating the table of chords, Hipparchus began with $60°$, so that the chord is equal to the radius, since we have an equilateral triangle. Thus $crd(60) = 57,18$ in sexagesimal or $3438'$ in minutes.

Now, for a $90°$ angle, the chord is equal to:

$$
\begin{aligned}
R\sqrt{2} &= 4862' \\
&= 81,2 \qquad (81 \times 60 + 2 \times 1)
\end{aligned}
$$

So $crd(90) = 81,2$ (in sexagesimal).

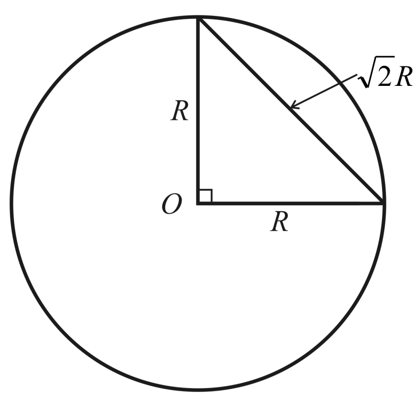

Figure 4: This figure shows the geometry for a chord subtending a right angle.

To calculate the chords of other angles, Hipparchus used the following geometric results:

Since the angle in a semi-circle is a right angle, we can use Pythagoras' Theorem to obtain

$$crd(180 - \alpha) = \sqrt{(2R)^2 - crd^2(\alpha)}.$$

Thus we only need to find the chords of angles up to $90°$.

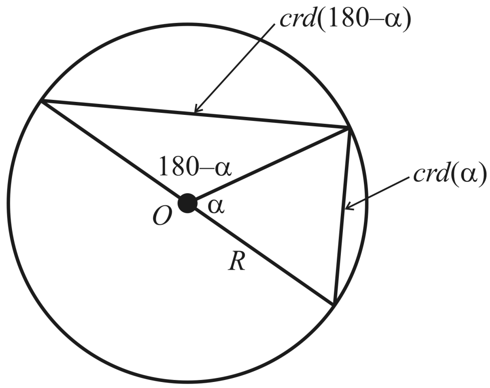

Figure 5: This figure shows two chords, one subtending an angle α and the second subtending the angle $180 - \alpha$.

Now, earlier we saw that $crd\,\alpha = 2R\sin\left(\frac{\alpha}{2}\right)$, and by using this result we can write

$$\begin{aligned} crd(180-\alpha) &= 2R\sin\left(\frac{180-\alpha)}{2}\right) \\ &= 2R\cos\left(\frac{\alpha}{2}\right). \end{aligned}$$

Again, in a circle of unit diameter, we see that the chord of the supplement of an angle, is the cosine of half the angle. Notice that,

$$\begin{aligned} crd(180-\alpha) &= \sqrt{(2R)^2 - crd^2(\alpha)} \\ 2R\cos\frac{\alpha}{2} &= \sqrt{(2R)^2 - (2R\sin\frac{\alpha}{2})^2} \\ (2R)^2\cos^2\frac{\alpha}{2} &= (2R)^2 - (2R)^2\sin^2\left(\frac{\alpha}{2}\right) \\ \text{i.e.} \qquad \sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} &= 1. \end{aligned}$$

Replacing α by 2α, we obtain the well-known result,

$$\sin^2\alpha + \cos^2\alpha = 1.$$

In order to calculate the table of chords, Hipparchus also found a formula for $crd\left(\frac{\alpha}{2}\right)$. From previous results,

$$crd\left(\frac{\alpha}{2}\right) = DC = BD, \text{ so } \angle BAD = \angle DAC.$$

Now, in $\Delta's\ AED$ and $ABD,\ AE = AB$, and AD is common, therefore $\Delta AED \equiv \Delta ABD$ (Side Angle Side).

Hence $BD = DE.$ and since $BD = DC$, we have $DC = DE.$

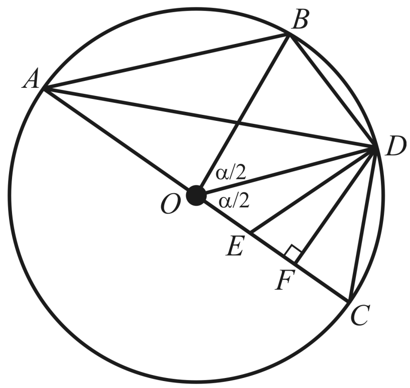

Figure 6: This figure shows the geometry that Hipparchus used to find a formula for the chord subtending $\alpha/2$.

If DF is drawn perpendicular to EC,

$$\text{then} \quad CF = \frac{1}{2}CE$$
$$= \frac{1}{2}(AC - AE)$$
$$= \frac{1}{2}(AC - AB),$$

but $AC = 2R$ and $AB = crd(180 - \alpha)$, therefore

$$CF = \frac{1}{2}(2R - crd(180 - \alpha)).$$

But $\Delta ACD|||\Delta DCF$ as they have one angle in common and are right angled,

$$\text{i.e.} \quad \frac{AC}{CD} = \frac{CD}{CF}$$
$$CD^2 = AC \cdot CF$$

Recall that $CD = crd\left(\frac{\alpha}{2}\right)$, $AC = 2R$ and $CF = \frac{1}{2}(2R - crd(180 - \alpha))$, therefore $crd^2\left(\frac{\alpha}{2}\right) = R(2R - crd(180 - \alpha))$.

Hipparchus used this formula to calculate chords of half angles.
In order to compare this to modern notation, substitute $crd(\frac{\alpha}{2}) = 2R\sin\frac{\alpha}{4}$ and $crd(180 - \alpha) = 2R\cos\left(\frac{\alpha}{2}\right)$. Then

$$\left(2R\sin\frac{\alpha}{4}\right)^2 = R\left(2R - 2R\cos\frac{\alpha}{2}\right)$$
$$\sin^2\frac{\alpha}{4} = \frac{1}{2}\left(1 - \cos\frac{\alpha}{2}\right),$$

and replacing α by 2α, we have the well-known formula:

$$\sin^2 \frac{\alpha}{2} = \frac{1}{2}(1 - \cos\alpha).$$

Here is a table of chords, written in sexagesimal.

Arcs	**Chords**	**Sixtieths**	**Arcs**	**Chords**	**Sixtieths**
$\frac{1}{2}$	0;31,25	0;1,2,50	6	6;16,49	0;1,2,44
1	1;2,50	0;1,2,50	47	47;51,0	0;0,57,34
$1\frac{1}{2}$	1;34,15	0;1,2,50	49	49;45,48	0;0,57,7
2	2;5,40	0;1,2,50	72	70;32,3	0;0,50,45
$2\frac{1}{2}$	2;37,4	0;1,2,48	80	77;8,5	0;0,48,3
3	3;8,28	0;1,2,48	108	97;4,56	0;0,36,50
4	4;11,16	0;1,2,47	120	103;55,23	0;0,31,18
$4\frac{1}{2}$	4;42,40	0;1,2,47	133	110;2,50	0;0,24,56

Many of the trigonometric formulae for the sum or difference of two angles, for multiples and for half-angles, can all be derived from a proposition known as *Ptolemy's Theorem*. It states that if the four vertices of a quadrilateral are concyclic, then the sum of the products of the opposite sides is equal to the product of the diagonals of the quadrilateral.

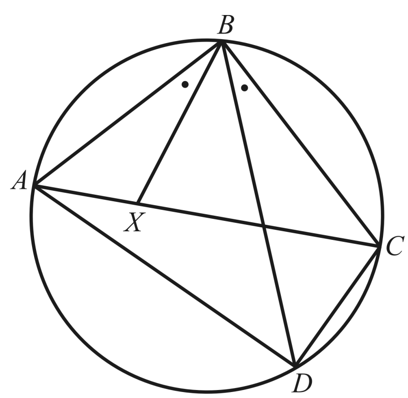

Figure 7: This figure shows the geometry for Ptolemy's Theorem in which the four vertices of a quadrilateral are concyclic.

Proof: In the cyclic quadrilateral $ABCD$, choose X on AC such that $\angle ABX = \angle DBC$.

Now $\angle ABD = \angle XBC$ (as $\angle XBD$ is common),

and $\angle BDA = \angle BCA$ (since angles standing on the same arc are equal).

In triangles ABD and XBC, $\angle BAD = \angle BXC$ (remaining angle in triangle), then $\Delta ABD ||| \Delta XBC$ (Angle Angle). Hence,

$$\frac{BD}{AD} = \frac{BC}{XC} \quad \text{(sides in same ratio)}$$

i.e. $AD \cdot BC = BD \cdot XC$.

Similarly, since $\angle BAC = \angle BDC$

$$\Delta ABX ||| \Delta DBC \quad \text{(Angle Angle).}$$

Hence, $\quad \dfrac{AB}{AX} = \dfrac{BD}{CD} \quad$ i.e. $\quad AB \cdot CD = BD \cdot AX.$
Summarising these results we have;

$$AB \cdot CD = BD \cdot AX$$
$$AD \cdot BC = BD \cdot XC$$

$$\begin{aligned} \text{Then} \quad AB \cdot CD + AD \cdot BC &= BD \cdot AX + BD \cdot XC \\ &= BD\,(AX + XC) \\ \text{but} \quad AX + XC &= AC, \text{ therefore} \\ AB \cdot CD + AD \cdot BC &= BD \cdot AC. \end{aligned}$$

This theorem can be applied to the calculation of chords, where either a side or a diagonal coincides with a diameter of the circumscribed circle.

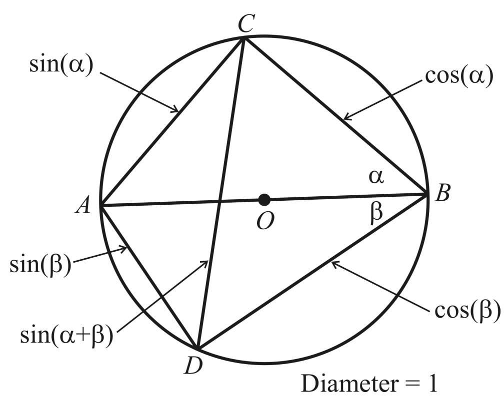

Figure 8: This figure shows the geometry for a cyclic quadrilateral with one side being the diameter of length one.

In the figure, AB is a diagonal and also a diameter of length one. Thus, as was shown previously, if $\angle CBA = \alpha$ and $\angle DBA = \beta$, then $AC = \sin\alpha, AD = \sin\beta, CB = \cos\alpha$ and $BD = \cos\beta$. Note that $CD = \sin(\alpha + \beta)$.

Now applying Ptolemy's Theorem, remembering $AB = 1$, we have

$$AB \cdot CD = AC \cdot BD + BC \cdot AD$$
$$\text{i.e.} \qquad \sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta.$$

Now suppose we take a cyclic quadrilateral with one side being the diameter of length one. Let $\angle ABC = \alpha,\ \angle ABD = \beta$ then, as before, $AC = \sin\alpha,\ AD = \sin\beta,\ CB = \cos\alpha,\ DB = \cos\beta$ and

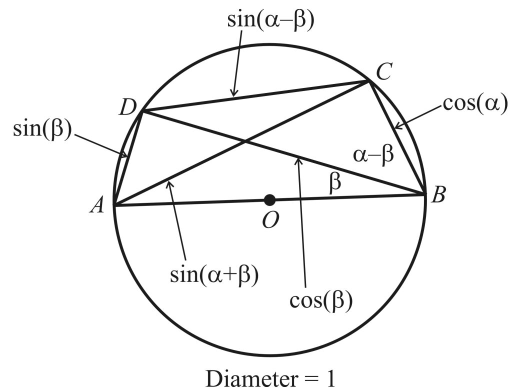

Figure 9: This figure shows the geometry for a second cyclic quadrilateral with one side being the diameter of length one.

$DC = \sin(\alpha - \beta)$.

Now, once again applying Ptolemy's Theorem, we have

$$\begin{aligned} AC \cdot BD &= DC \cdot AB + AD \cdot BC \\ \sin\alpha\ \cos\beta &= \sin(\alpha - \beta)\ .1 + \sin\beta\ \cos\alpha \\ \text{i.e.}\quad \sin(\alpha - \beta) &= \sin\alpha\ \cos\beta - \cos\alpha\ \sin\beta. \end{aligned}$$

Finally, we again take a cyclic quadrilateral with one side as a diameter, and we choose $CD = BD$. Then if $\angle CAB = \theta$, we have $\angle CAD = \angle DAB = \frac{\theta}{2}$, (since equal arcs subtend equal angles). We can again express each of the sides in terms of trigonometric ratios and apply Ptolemy's Theorem to obtain:

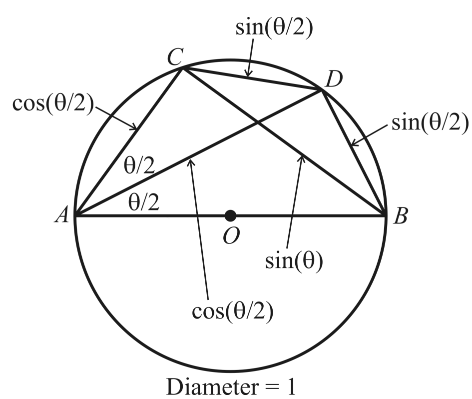

Figure 10: This figure shows the geometry for the cyclic quadrilateral used to establish the dichotomy formula.

$$\sin\frac{\theta}{2} + \sin\frac{\theta}{2}\,\cos\theta = \cos\frac{\theta}{2}\,\sin\theta.$$

Dividing by $\sin\frac{\theta}{2}$, we have $1+\cos\theta = \cot\frac{\theta}{2}\,\sin\theta.$

$$\text{Hence}\quad \tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} \times \left(\frac{1-\cos\theta}{1-\cos\theta}\right) = \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$$

$$= \frac{1-\cos\theta}{\sin\theta}$$

$$\therefore\ \tan\frac{\theta}{2} = \text{cosec}\,\theta - \cot\theta.$$

This is referred to as the dichotomy formula, and some form of it was used by Archimedes in the computation of π some 400 years prior to Ptolemy.

As mentioned earlier, the origins of trigonometry lie in the world of astronomy and spherical triangles. It was Regiomontanus who introduced trigonometry into a form that we would recognise today. Born Johann Müller, Regiomontanus took his name from the latinized form of his hometown, Königsberg, 'King's Mountain'. He was born in 1436 and died in 1476. Early on in his life he studied at home, then he was sent to Vienna at age 12 where he received his Bachelor's degree at age 15. His most influential work was his 'De triangulis omnimodis' (*On triangles of every kind*) which was a work in five parts. In part one of the book he introduces the sine function to solve a right angled triangle; and in book two he introduces trigonometry proper with the Law of Sines.

The formula for the area of a triangle given two sides and the included angle ($A = \frac{1}{2}ab\sin C$) also appears here for the first time. It was written as:

> If the area of a triangle is given together with the rectangular product of the two sides, then either the angle opposite the base becomes known, or (that angle) together with (its) known (exterior) equals two right angles.

He also deals with spherical geometry and trigonometry in the remaining three books, but never used the tangent function, although it was obvious that he was familiar with it.

The following problem was posed by Regiomontanus in 1471:

At what point on the ground does a perpendicular suspended rod appear largest (i.e. subtends the greatest visual angle)?
In other words, if the rod AB at height b above the ground above 0 is viewed from a point X, at what distance will AB subtend the largest angle θ?

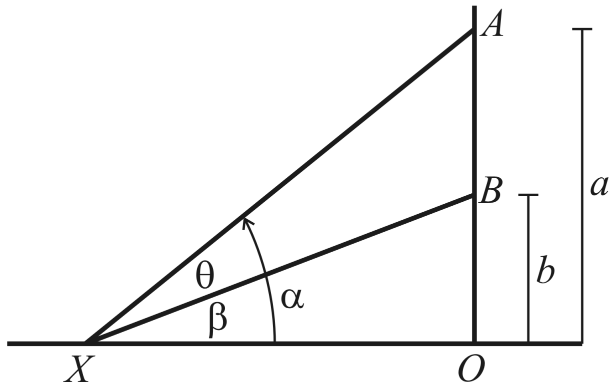

Figure 11: This figure shows the geometry for Regiomontanus' problem.

In the diagram,

$$\begin{aligned}\cot\theta &= \cot(\alpha-\beta)\\ &= \frac{1}{\tan(\alpha-\beta)}\\ &= \frac{1+\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}\\ &= \frac{1+\frac{1}{\cot\alpha\cot\beta}}{\frac{1}{\cot\alpha}-\frac{1}{\cot\beta}}\\ &= \frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}\end{aligned}$$

Let $OX = x$, then $\cot\alpha = \dfrac{x}{a}$ and $\cot\beta = \dfrac{x}{b}$, therefore

$$\begin{aligned}\cot\theta &= \frac{\left(\frac{x}{a}\right)\left(\frac{x}{b}\right)+1}{\left(\frac{x}{b}\right)-\left(\frac{x}{a}\right)} = \frac{x^2+ab}{x(a-b)}\\ &= \frac{x}{a-b}+\frac{ab}{(a-b)x}.\end{aligned}$$

Let $u = \dfrac{x}{a-b}$ and $v = \dfrac{ab}{(a-b)x}$. The arithmo-geometric inequality states that if $u, v \geq 0$ then $u+v \geq 2\sqrt{uv}$. Applying this inequality, we have:

$$\cot\theta = u+v \geq 2\sqrt{uv} = \frac{2\sqrt{ab}}{a-b}.$$

To maximise $\theta, \cot\theta$ needs to be minimised, this occurs when $u = v$, that is, when $\dfrac{x}{a-b} = \dfrac{ab}{(a-b)x} \Rightarrow x = \sqrt{ab}$. Thus, the point x is to be located at a distance equal to the geometric mean of the altitudes of the endpoints of the rod, measured horizontally from the foot of the rod.

The origin of the term 'sine' seems to have originally come from India and was adopted by Arab mathematicians. It was originally referred to as 'jya-ardha' which meant 'chord-half', and was at times shortened to 'jiva'. The Arab mathematicians phonetically derived the meaningless word 'jiba', and it was written in Arabic without vowels as 'jb'. This was later interpreted as 'jaib' which means 'breast'. After King Alfonso of Castile conquered Toledo in 1085 and captured a large library including many Arab manuscripts, scholars were hired to translate these books into Latin. The Latin word for 'breast' is 'sinus', which also means 'bay' or 'gulf'. This Latin word then became 'sine'. Some sources suggest that the Latin term 'sine' was introduced by Robert of Chester (1145), while others suggest it was introduced by Gherardo of Cremona (1150). In English, it seems the earliest use of sine was in 1593 by Thomas Fale.

With the need to find the sine of the complementary angle, cosine was introduced by Edmund Günter in 1620. It was originally written as 'co.sinus', short for 'complementi sinus'. Co.sinus was later modified to cosinus by John Newton (1658).

The word tangent was introduced by Thomas Fincke in (1583) from the Latin word 'tangere' which means 'to touch'. Francois Vieta (1593) was not comfortable with the word tangent because of its meaning in geometry, and so he used the term 'sinus foecundarum'.

Secant, introduced by Thomas Fincke, comes from the Latin 'secare', which means 'to cut' (1583). Once again, Vieta (1593) was not comfortable with this as it could have been confused with the geometric term, hence he used 'transsinuosa'. The cosine of an angle is the sine of the complementary angle, the cotangent of an angle is the tangent of the complementary angle and cosecant is the secant of the complementary angle.

References:

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www.pup.princeton.edu/books/maor/sidebar_c.pdf[3]

http://www.hps.cam.ac.uk/starry/hipparchus.html

A thank you to Peter Brown of the UNSW who has helped in the writing of this article.

[3]Editorial note, February 2014: this link is now dead. But see also http://press.princeton.edu/titles/10065.html