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Problems Section

You are invited to submit solutions to any or all of the following problems, accompanied by your name, school and year. Solutions to these problems will appear in the next issue of *Parabola*, and if received in time your solution(s) may be used.

Q1211. Solve

$$(2+\sqrt{2})^{\sin^2 x} - (2+\sqrt{2})^{\cos^2 x} + (2-\sqrt{2})^{\cos 2x} = \left(1+\frac{1}{\sqrt{2}}\right)^{\cos 2x}$$

Q1212. Assume that the equation

$$x^4 + ax^3 + bx^2 + cx + 1 = 0$$

has at least one solution. Prove that

$$a^2 + b^2 + c^2 \geq \frac{4}{3}.$$

Q1213. (submitted by Ildar Gaisin, Year 11, All Saints Anglican School, edited by Editor)

Let n be a positive integer and α be a real number such that α is not a multiple of π, and that the following numbers are all defined

$$\tan(\alpha),\ \tan(2\alpha),\ \tan(2^2\alpha),\ \ldots,\ \tan(2^{n-1}\alpha).$$

Find the sum of all the products of the squares of these numbers taking k distinct numbers at a time, where k takes the values from 1 to n. For example, if $n = 3$ there are 3 numbers $\tan(\alpha)$, $\tan(2\alpha)$ and $\tan(2^2\alpha)$, and the sum to be found is

$$\begin{aligned} S &= \tan^2(\alpha) + \tan^2(2\alpha) + \tan^2(2^2\alpha) \\ &\quad + \tan^2(\alpha)\tan^2(2\alpha) + \tan^2(\alpha)\tan^2(2^2\alpha) + \tan^2(2\alpha)\tan^2(2^2\alpha) \\ &\quad + \tan^2(\alpha)\tan^2(2\alpha)\tan^2(2^2\alpha). \end{aligned}$$

Q1214. Four towns are located at the vertices of a square of side length 10 km. Roads are to be built to connect these towns so that people can get from one town to the other towns.

1. Resources are enough to build only up to 28 km of roads. Is it possible to achieve the goal set?

2. Design the shortest roads possible.

Q1215. Prove that if a, b and c are three sides of a triangle whose area is S then

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3}.$$

When does the equality occur?

Q1216. Prove that it is not possible to place 5 balls of the same size so that each one touches the other four.

Q1217. (submitted by Julius Guest, East Bentleigh, Victoria)

Prove that in any triangle ΔABC the following is true

$$a \cos B \cos C + b \cos C \cos A + c \cos A \cos B = 2\Delta \frac{\sin A}{a},$$

where Δ is the area of the triangle, and a, b and c are the lengths of sides opposite the vertices A, B and C, respectively.

Q1218. Let the area S and one angle α of a triangle be given. Determine the lengths of 2 sides such that the side opposite α is as short as possible.

Q1219. (submitted by Julius Guest, East Bentleigh, Victoria)

A triangle ΔABC satisfies

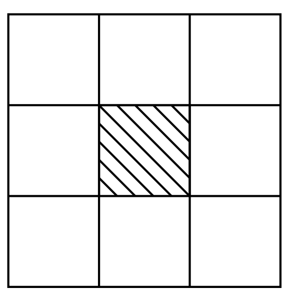
$$b + c = 2a,$$

where a, b and c are the lengths of sides opposite A, B and C, respectively. Prove that

$$4\Delta = 3a^2 \tan(A/2),$$

where Δ is the area of the triangle.

Q1220. A square of unit length is divided into 9 equal squares as in the figure. The central square is removed. Repeat the same process n times for the remaining squares.



1. How many squares of side length $1/3^n$ remain?
2. What is the sum of the areas of the removed squares as n grows to infinity?