Parabola Volume 44, Issue 3 (2008)

Solutions to Problems 1271-1280

Q1271 (suggested by Julius Guest, Victoria)

Solve simultaneously

$$x^2 + xy + y^2 = 189$$
$$x - \sqrt{xy} + y = 9.$$

ANS: (suggested by Julius Guest, Victoria, and Keith Anker, Victoria) Let $p = x+y$ and $q = \sqrt{xy}$. Then it follows from the given equations that $p - q = 9$ and $p^2 - q^2 = 189$, implying $p + q = 21$. Hence $p = 15$ and $q = 6$. From $x + y = 15$ and $xy = 36$ we deduce $x = 3$ and $y = 12$ or $x = 12$ and $y = 3$.

Q1272 Find all whole numbers such that when the third digit is deleted, the resulting number divides the original one. (For example 34 divides 340.)

ANS: Let $x = a_n a_{n-1} a_{n-2} \cdots a_1 a_0$ be such a number. Note that x has at least 3 digits, that is, $n \geq 2$. Then x can be written as

$$x = a_n \times 10^n + a_{n-1} \times 10^{n-1} + a_{n-2} \times 10^{n-2} + \cdots + a_1 \times 10 + a_0.$$

Let y be the number obtained from x by deleting the third digit, that is,

$$\begin{aligned} y &= a_n a_{n-1} a_{n-3} a_{n-4} \cdots a_1 a_0 \\ &= a_n \times 10^{n-1} + a_{n-1} \times 10^{n-2} + a_{n-3} \times 10^{n-3} + \cdots + a_1 \times 10 + a_0. \end{aligned}$$

Then

$$\begin{aligned} x - 10y = (a_{n-2} - a_{n-3}) \times 10^{n-2} + (a_{n-3} - a_{n-4}) \times 10^{n-3} + \cdots \\ + (a_2 - a_1) \times 10^2 + (a_1 - a_0) \times 10 + a_0, \end{aligned}$$

so that, since $0 \leq a_k \leq 9$ for all $k = 0, \ldots, n$,

$$\begin{aligned} |x - 10y| &\leq 9 \times 10^{n-2} + 9 \times 10^{n-3} + \cdots + 9 \times 10 + 9 \\ &= 10^{n-1} - 1 < y. \end{aligned}$$

On the other hand, since y divides x, it also divides $x - 10y$. Hence $x - 10y = 0$, that is, $x = 10y$. By comparing the digits of x and $10y$ we infer

$$0 = a_0 = a_1 = a_2 = \cdots = a_{n-2}.$$

Therefore, all numbers x are of the form $a_n a_{n-1} 0 \cdots 0$.

Q1273 Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 1$ and

$$f(x-y) = \frac{1}{2}f(a-y)f(x) + \frac{1}{2}f(a-x)f(y) \tag{1}$$

for all $x, y \in \mathbb{R}$ and a given $a \in \mathbb{R}$.

ANS: Put $x = y$ in (1). Then

$$f(0) = \frac{1}{2}f(a-x)f(x) + \frac{1}{2}f(a-x)f(x),$$

so that

$$f(x) = \frac{1}{f(a-x)}. \tag{2}$$

Note that $f(a) = 1/f(0) = 1$ follows from this. Now consider $y = 0$. Then

$$f(x) = \frac{1}{2}f(a)f(x) + \frac{1}{2}f(a-x)f(0),$$

and hence

$$f(x) = \frac{1}{2}f(x) + \frac{1}{2}f(a-x),$$

so that

$$f(x) = f(a-x) \quad \text{for all } x \in \mathbb{R}. \tag{3}$$

Identities (2) and (3) imply

$$f^2(x) = 1 \quad \text{for all } x \in \mathbb{R}. \tag{4}$$

Substituting (3) into (1) yields

$$f(x-y) = f(x)f(y).$$

Letting $y = -x$ we deduce

$$f(2x) = f(x)f(-x) \quad \text{for all } x \in \mathbb{R}. \tag{5}$$

On the other hand, letting $x = 0$ in (1) we obtain

$$f(-y) = \frac{1}{2}f(a-y) + \frac{1}{2}f(y),$$

which, together with (3), yields

$$f(-y) = f(y) \quad \text{for all } y \in \mathbb{R}.$$

Thus, (5) and (4) give

$$f(2x) = f^2(x) = 1 \quad \text{for all } x \in \mathbb{R},$$

that is, $f(x) = 1$ for all $x \in \mathbb{R}$. It is clear that $f(x) = 1$ for all $x \in \mathbb{R}$ satisfies (1), so the only function which satisfies (1) is the constant function 1.

Q1274 Does there exist a natural number n such that the fractional part of the number $(2+\sqrt{2})^n$ exceeds 0.999999?

ANS: Let

$$A = (2+\sqrt{2})^n, \quad B = (2-\sqrt{2})^n \quad \text{and} \quad C = A + B.$$

It is easy to see that $0 < B < 1$. By using the binomial formula we can prove that C is an integer. Indeed, all odd powers of $\sqrt{2}$ cancel out, whereas all of its even powers are integers. By writing A as a sum of an integer I and a fractional part F (that is, $0 < F < 1$) we have

$$C = I + F + B,$$

that is,

$$C - I = F + B.$$

The left-hand side is an integer, and the right-hand side is less than 2, so $C - I = F + B = 1$, implying $F = 1 - B$. Now $F > 0.999999$ if and only if $B < 10^{-6}$, which happens when $n \geq 26$.

Q1275 Show that, for all integers $n > 2$,

$$2^{n(n-1)/2} > n!$$

ANS: Since

$$1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2},$$

we have

$$\begin{aligned} 2^{n(n-1)/2} &= 2^{1+2+3+\cdots+(n-1)} \\ &= 2^1 \cdot 2^2 \cdot 2^3 \cdots 2^{n-1}. \end{aligned}$$

Since

$$2^{n-1} > n \quad \text{for all } n \geq 3 \tag{6}$$

we have

$$2^{n(n-1)/2} > 2 \cdot 3 \cdot 4 \cdots n = n!$$

Inequality (6) can be shown by induction, or by noting that

$$\begin{aligned} 2^{n-1} &= (1+1)^{n-1} \\ &= \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \cdots + \binom{n-1}{n-1} \\ &> \binom{n-1}{0} + \binom{n-1}{1} = 1 + n - 1 = n. \end{aligned}$$

Q1276 If b_1, b_2, c_1 and c_2 are real numbers such that

$$b_1 b_2 = 2(c_1 + c_2),$$

show that at least one of the equations

$$x^2 + b_1 x + c_1 = 0$$
$$x^2 + b_2 x + c_2 = 0$$

has two real roots.

ANS: (suggested by Keith Anker, Victoria)

Suppose that both equations don't have real roots. Then

$$b_1^2 - 4c_1 < 0 \quad \text{and} \quad b_2^2 - 4c_2 < 0,$$

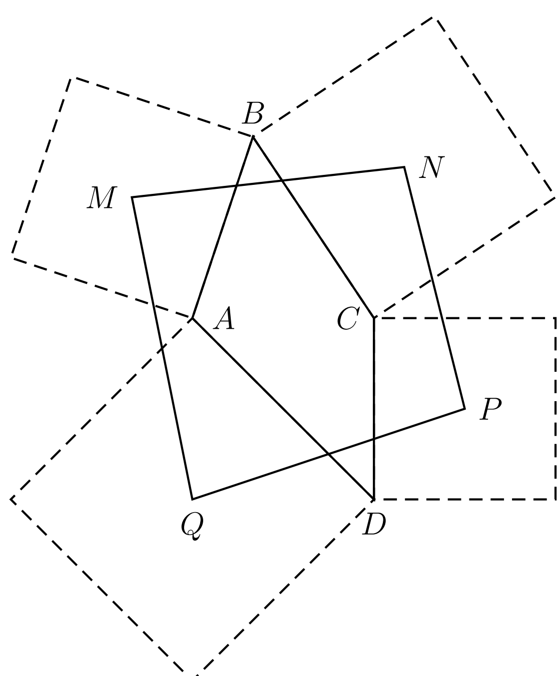
which imply

$$b_1^2 + b_2^2 < 4(c_1 + c_2) = 2b_1 b_2.$$

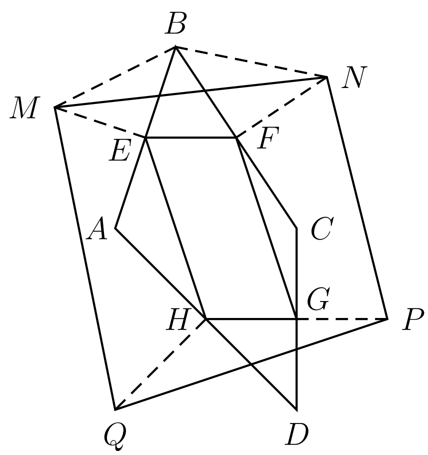
Hence, $(b_1 - b_2)^2 < 0$, which is false. So at least one equation has two real roots (which may be double roots).

Q1277 Four squares are constructed from 4 sides of a convex quadrilateral $ABCD$ such that the squares are all in the exterior of $ABCD$. Let S be the area of $ABCD$. Show that the area S' of the quadrilateral formed by the centres of the squares satisfies $S' \geq 2S$. When does equality occur?

ANS: Let M, N, P and Q be the centres of the squares formed, respectively, from AB, BC, CD and DA.



Let E, F, G and H be the midpoints of AB, BC, CD and DA, respectively.



Then

$$\begin{aligned} S' &= S_{MNPQ} \\ &= S_{EFGH} + S_{EMNF} + S_{FNPG} + S_{HGPQ} + S_{EHQM} \end{aligned} \tag{7}$$

Now consider S_{EMNF}. Let $\alpha = \angle MBN$. We have

$$\begin{aligned} S_{EMNF} &= S_{BFE} + S_{BEM} + S_{BNF} \pm S_{BNM} \\ &= S_{BFE} + \frac{1}{2} BE \cdot ME + \frac{1}{2} BF \cdot NF \pm \frac{1}{2} BM \cdot BN \cdot \sin\alpha \\ &= S_{BFE} + \frac{AB^2}{8} + \frac{BC^2}{8} \pm \frac{1}{4} AB \cdot BC \cdot \sin\alpha, \end{aligned} \tag{8}$$

with $+$ when B is inside $MNPQ$ and $-$ when B is outside $MNPQ$. Let $\beta = \angle ABC$. Then B is outside $MNPQ$ when $0 < \beta < \pi/2$, B is inside $MNPQ$ when $\pi/2 < \beta < \pi$, and B is on MN when $\beta = \pi/2$. On the other hand,

$$\alpha = \begin{cases} \frac{3\pi}{2} - \beta & \text{if } B \text{ is inside } MNPQ \\ \beta + \frac{\pi}{2} & \text{if } B \text{ is outside } MNPQ \\ \pi & \text{if } B \text{ is on } MN, \end{cases}$$

so that

$$\sin\alpha = \begin{cases} -\cos\beta & \text{if } B \text{ is inside } MNPQ \\ \cos\beta & \text{if } B \text{ is outside } MNPQ \\ 0 & \text{if } B \text{ is on } MN. \end{cases}$$

Therefore, it follows from (8) that

$$\begin{aligned} S_{EMNF} &= S_{BFE} + \frac{AB^2}{8} + \frac{BC^2}{8} - \frac{1}{4} AB \cdot BC \cdot \cos\beta \\ &= S_{BFE} + \frac{1}{8} AC^2. \end{aligned} \tag{9}$$

Similarly, we have

$$\begin{aligned} S_{FNPG} &= S_{CGF} + \frac{1}{8}BD^2, \\ S_{HGPQ} &= S_{DHG} + \frac{1}{8}AC^2, \\ S_{EHQM} &= S_{AEH} + \frac{1}{8}BD^2. \end{aligned} \tag{10}$$

It follows from (7), (9) and (10) that

$$\begin{aligned} S' &= S_{EFGH} + S_{BFE} + S_{CGF} + S_{DHG} + S_{AEH} + \frac{1}{4}(AC^2 + BD^2) \\ &= S_{ABCD} + \frac{1}{4}(AC^2 + BD^2) \geq S + \frac{1}{2}AC \cdot BD \\ &\geq S + \frac{1}{2}AC \cdot BD \cdot \sin\gamma = 2S, \end{aligned}$$

where γ is the (smaller) angle formed by AC and BD. Equality occurs when $AC = BD$ and $\sin\gamma = 1$, i.e., $AC = BD$ and $\gamma = \pi/2$. In other words, $S' = 2S$ if and only if the diagonals AC and BD of $ABCD$ are equal and perpendicular.

Q1278 Find all prime numbers p such that $(2^{p-1} - 1)/p$ is a perfect square.

ANS: It is to check that $p = 3$ and $p = 7$ satisfy the required condition. We will prove that they are the only ones.

First we note that $p \neq 2$. Now if $2^{p-1} - 1 = pm^2$, then

$$\left(2^{(p-1)/2} - 1\right)\left(2^{(p-1)/2} + 1\right) = pm^2.$$

Noting that $p - 1$ is even we deduce that $2^{(p-1)/2} - 1$ and $2^{(p-1)/2} + 1$ are integers, and that they differ by 2. As they are both odd, they are relatively prime. So one of them equals m_1^2 and the other equals pm_2^2 where $m = m_1 m_2$.

Note that m_1 is odd. We will prove that $2^{(p-1)/2} - 1$ equals an odd square (namely, m_1^2) only when $p = 3$, and $2^{(p-1)/2} + 1$ equals an odd square only when $p = 7$.

If $p > 3$ then $2^{(p-1)/2} - 1 \equiv -1 (\mathrm{mod}\ 4)$. So it cannot be a square of an odd number (which has to be $1 (\mathrm{mod}\ 4)$).

If $2^{(p-1)/2} + 1$ is the square of an odd, then

$$2^{(p-1)/2} + 1 = (2k + 1)^2 = 4k(k + 1) + 1$$

for some integer k. This is equivalent to

$$k(k + 1) = 2^{(p-5)/2}.$$

The above identity only holds when $p = 7$ (i.e., $k = 1$). Indeed, if $p \neq 7$, then the left-hand side (which is a product of two consecutive integers) must contain an odd prime factor, whereas the right-hand side has only 2 as prime factor. So $2^{(p-1)/2} + 1$ is the square of an odd only when $p = 7$.

Q1279 One hour after leaving home, a car broke down and had to travel the remaining trip at $3/5$ of its original speed. The car arrived 2 hours late. If the incident had occurred 50 km further along, the care would have arrived 40 minutes sooner. What is the original speed of the car?

ANS: Let the original speed be x km/h. Then the speed after the break-down is $3x/5$ km/h. Since travelling at x km/h over a distance of 50 km takes 2/3 hour (or 40 minutes) less time than travelling at $3x/5$ km/h, we have

$$\frac{2}{3} + \frac{50}{x} = \frac{50}{3x/5}.$$

Solving this equation yields $x = 50$.

Q1280 Show that in a triangle the median corresponding to the longest side is shorter than that side.

ANS: In ΔABC assume that $c \leq b \leq a$. Then $\angle AMC \geq 90°$, so that it is the largest angle in ΔAMC. Hence, $m_a \leq b \leq a$.