Problem 1
Find
\[ S = 1 + 11 + 111 + \cdots + \underbrace{11\cdots1}_{2014 \text{ digits}}. \]

Proof. [Solution] By sum of the geometric progression formula:
\[
\underbrace{11\cdots1}_k = 1 + 10 + 10^2 + \cdots + 10^{k-1} = \frac{10^k - 1}{10 - 1} = \frac{1}{9}(10^k - 1),
\]
so we compute
\[
S = \sum_{k=1}^{2014} \frac{1}{9}(10^k - 1)
= \frac{1}{9} \left[ \sum_{k=1}^{2014} 10^k - \sum_{k=1}^{2014} 1 \right]
= \frac{1}{9} \left[ 10(1 + 10 + \cdots + 10^{2013}) - 2014 \right]
= \frac{1}{9} \left[ \frac{10(10^{2014} - 1)}{9} - 2014 \right] \quad \text{(sum of a GP)}
= \frac{1}{81} \left[ 10^{2015} - 10 \cdot 2014 \right]
= \frac{1}{81} \left[ 10^{2015} - 18136 \right]
\]

We can find the decimal representation of \( S \) as follows. By setting
\[ x = 10^{2015} - 18136, \]
we see that
\[ S = \frac{x}{81}. \]
On the other hand, writing out \( x \) in base 10 we find,

\[
x = \underbrace{99\ldots9}_{2010 \text{ digits}} \quad 81864.
\]

Hence,

\[
\frac{x}{9} = \underbrace{11\ldots1}_{2010 \text{ digits}} \quad 09096.
\]

We now divide by 9 again. Since

\[
\underbrace{11\ldots1}_{9 \text{ digits}} = 9 \times (012345679)
\]

then extracting groups of 9 1s in \( x/9 \), and dividing by 9 and as

\[
2010 = 9 \times 223 + 3
\]

then \( S = x/81 \) in base 10 consists of 223 groups of 012345679 followed by

\[
11109096 \div 9 = 01234344.
\]

Hence,

\[
S = \underbrace{(012345679)(012345679)\cdots(012345679)}_{223 \text{ times}} \quad 01234344
\]

\[
= \underbrace{(123456790)(123456790)\cdots(123456790)}_{223 \text{ times}} \quad 1234344.
\]

**Problem 2**

Let \( a_1, a_2, \ldots, a_n \) be real numbers such that \( a_1 + \cdots + a_n = 1 \). Prove that

\[
a_1^2 + \cdots + a_n^2 \geq \frac{1}{n}.
\]
Proof. [Solution 1] A possible solution can be

\[
0 \leq \sum_{i=1}^{n} \left( a_i - \frac{1}{n} \right)^2
= \sum_{i=1}^{n} \left( \frac{a_i^2}{n} - \frac{2a_i}{n} + \frac{1}{n^2} \right)
= \sum_{i=1}^{n} a_i^2 - \frac{2}{n} \sum_{i=1}^{n} a_i + \frac{1}{n^2} \sum_{i=1}^{n} 1
= \sum_{i=1}^{n} a_i^2 - \frac{2}{n} \cdot 1 + \frac{1}{n^2} \cdot n
= \sum_{i=1}^{n} a_i^2 - \frac{1}{n}
= \sum_{i=1}^{n} a_i^2 - \frac{1}{n}.
\]

Hence,

\[
\frac{1}{n} \leq \sum_{i=1}^{n} a_i^2.
\]

\[\square\]

Proof. [Solution 2] A similar solution is to shift

\[a_i = \frac{1}{n} + x_i \text{ for } 1 \leq i \leq n\]

and then compute

\[
1 = \sum_{i=1}^{n} a_i
= \sum_{i=1}^{n} \frac{1}{n} + \sum_{i=1}^{n} x_i
= \frac{1}{n} \cdot n + \sum_{i=1}^{n} x_i
= 1 + \sum_{i=1}^{n} x_i
\]

so

\[
\sum_{i=1}^{n} x_i = 0.
\]
Hence,
\[
\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} \left( \frac{1}{n} + x_i \right)^2 \\
= \sum_{i=1}^{n} \left( \frac{1}{n^2} + \frac{2x_i}{n} + x_i^2 \right) \\
= \left( \sum_{i=1}^{n} \frac{1}{n^2} \right) + \frac{2}{n} \left( \sum_{i=1}^{n} x_i \right) + \left( \sum_{i=1}^{n} x_i^2 \right) \\
= n \cdot \frac{1}{n^2} + \frac{2}{n} \cdot 0 + \left( \sum_{i=1}^{n} x_i^2 \right) \\
= \frac{1}{n} + \left( \sum_{i=1}^{n} x_i^2 \right) \\
\geq \frac{1}{n}.
\]

\[\Box\]

**Proof.** [Solution 3] Recall the Cauchy-Schwartz inequality:
\[
\vec{a} \cdot \vec{b} \leq ||\vec{a}|| ||\vec{b}||,
\]
where
\[
\vec{a} = (a_1, a_2, \ldots, a_n) \text{ and } \vec{b} = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right).
\]

Hence,
\[
\frac{1}{n} = \frac{a_1 + a_2 + \cdots + a_n}{n} \\
= \vec{a} \cdot \vec{b} \\
\leq ||\vec{a}|| ||\vec{b}|| \\
= \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} \frac{1}{n^2}} \\
= \sqrt{\frac{1}{n} \sum_{i=1}^{n} a_i^2}
\]
from which the result follows. \[\Box\]

**Problem 3**
Find all possible decimal digits you can use to fill places marked with an asterisk \(*\), so that the following identity holds
\[
*00* = (**)^2.
\]
Proof. [Solution] We want integers \( x \) with

\[
100 \leq x \leq 999
\]

such that for some integer \( a, 1 \leq a \leq 9, \)

\[
100000a \leq x^2 < 100000a + 1000
\]

\[
i.e. \quad 100\sqrt{10a} \leq x < 10\sqrt{1000a + 10}.
\]

Rounding to 1 decimal place, a calculator gives the following results with the 7 solutions to \( x \) and \( x^2 \) in the last two columns:

<table>
<thead>
<tr>
<th>( a )</th>
<th>( 100\sqrt{10a} )</th>
<th>( 10\sqrt{1000a + 10} )</th>
<th>( x )</th>
<th>( x^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>316.2</td>
<td>317.8</td>
<td>317</td>
<td>100489</td>
</tr>
<tr>
<td>2</td>
<td>447.2</td>
<td>448.3</td>
<td>448</td>
<td>200704</td>
</tr>
<tr>
<td>3</td>
<td>547.7</td>
<td>548.6</td>
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<td>632.4</td>
<td>633.2</td>
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<tr>
<td>5</td>
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<td>707.8</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>6</td>
<td>774.5</td>
<td>775.2</td>
<td>775</td>
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</tr>
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<tr>
<td>8</td>
<td>894.4</td>
<td>894.9</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>9</td>
<td>948.6</td>
<td>949.2</td>
<td>949</td>
<td>900601</td>
</tr>
</tbody>
</table>

\[\square\]

Problem 4
Five speakers A, B, C, D and E take part in a conference. Find the total number of ways to organise the programme so that

a) A speaks immediately before B;

b) B does not speak before A.

Proof. [Solution] For a)

\[
\text{No.} = 4 \times 1 \times 3! \\
\downarrow \\
\text{Choose a position for A} \\
\downarrow \\
\text{Place B in next position} \\
\downarrow \\
\text{Fill remaining positions}
\]

\[= 4 \times 6 = 24\]
For b)

\[
\text{No.} = \sum_{j=1}^{4} 1 \times (5 - j) \times 3!
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

Place A in position \( j \)
Choose a later position for B
Fill remaining positions

\[
= (4 + 3 + 2 + 1) \times 6
\]

\[
= 10 \times 6 = 60
\]

\[\square\]

Problem 5

a) Prove that the radius of the inscribed circle to the triangle \( \triangle ABC \) is given by

\[
r = \frac{2S}{AB + BC + AC},
\]

where \( S \) is the total area of the triangle \( \triangle ABC \).

b) In a right-angled triangle, we draw the altitude onto the hypotenuse. This process is repeated in the two smaller right-angled triangles so formed and the process is then continued 2014 times, as shown in the diagram. A circle is inscribed in each of the resulting \( 2^{2014} \) triangles. Find the total area of these circles.
Proof. [Solution, part a] Let $AB = c$, $AC = b$, $BC = a$ (lengths). Let the incircle to $\triangle ABC$ meet the sides at $P, Q, R$ as shown. Since the sides of $\triangle ABC$ are tangent to the in-circle, then $PO$, $QO$ and $RO$ are perpendicular to the respective sides $AB$, $AC$ and $BC$. Hence $PO$ is an altitude for $\triangle AOB$, $QO$ is an altitude for $\triangle AOC$ and $RO$ is an altitude for $\triangle BOC$, and these altitudes have length $r$, the radius of the incircle. Now $\triangle ABC$ is partitioned into sub-triangles $\triangle AOB$, $\triangle BOC$, $\triangle AOC$. Hence

$$S = \text{Area}(\triangle ABC) = \text{Area}(\triangle AOB) + \text{Area}(\triangle BOC) + \text{Area}(\triangle AOC)$$

$$= \frac{1}{2} rc + \frac{1}{2} ra + \frac{1}{2} rb$$

$$= \frac{r}{2} (c + b + a).$$

Therefore,

$$r = \frac{2S}{a + b + c} = \frac{2S}{BC + AC + BC}.$$ 

Proof. [Solution, part b] Let $\triangle ABC$ be a right-angled triangle, w.l.o.g. let $\angle ACB$ be a right angle. Let $CQ$ be an altitude to $\triangle ABC$. 

7
Let \( S, r, S_1, r_1 \) and \( S_2, r_2 \) be the area of the triangle and the radius of the inscribed circle for \( \triangle ABC, \triangle BQC \) and \( \triangle CQA \) respectively.

Now \( \triangle BQC \parallel \| \triangle BCA \parallel \| \triangle CQA \) (with corresponding sides in order) by “AAA” by common angles, right angles and angle sum of a triangle.

Since \( \triangle BQC \parallel \| \triangle BCA \) then
\[
\frac{x}{a} = \frac{z}{b} = \frac{a}{c}
\]
so
\[
x = a \cdot \frac{a}{c}, \quad z = b \cdot \frac{a}{c}
\]
and so
\[
S_1 = \frac{1}{2} x z = \frac{1}{2} a b \left( \frac{a}{c} \right)^2 = S \left( \frac{a}{c} \right)^2
\]
and
\[
a + x + z = a + a \left( \frac{a}{c} \right) + b \left( \frac{a}{c} \right) = (c + a + b) \left( \frac{a}{c} \right).
\]

Therefore,
\[
r_1 = \frac{2 S_1}{a + x + z} = \frac{2 S \left( \frac{a}{c} \right)^2}{(c + a + b) \left( \frac{a}{c} \right)} = r \left( \frac{a}{c} \right).
\]

Similarly, as \( \triangle CQA \parallel \| \triangle BCA \),
\[
\frac{y}{b} = \frac{z}{a} = \frac{b}{c}
\]
so
\[
y = b \cdot \frac{b}{c}, \quad z = a \cdot \frac{b}{c}
\]
and so
\[
S_2 = \frac{1}{2} y z = \frac{1}{2} b a \left( \frac{b}{c} \right)^2 = S \left( \frac{b}{c} \right)^2
\]
and
\[
b + z + y = b + a \left( \frac{b}{c} \right) + b \left( \frac{b}{c} \right) = (c + a + b) \left( \frac{b}{c} \right).
\]
Consequently,

\[ r^2 = \frac{2S}{b+z+y} = \frac{2S \left( \frac{b}{c} \right)^2}{(c+a+b) \left( \frac{b}{c} \right)} = r \left( \frac{b}{c} \right). \]

Hence the sum of the areas of the inscribed circles in \( \triangle BCQ \) and \( \triangle ACQ \) is

\[
\pi r_1^2 + \pi r_2^2 = \pi r^2 \left( \frac{a}{c} \right)^2 + \pi r^2 \left( \frac{b}{c} \right)^2
= \frac{\pi r^2}{c^2} (a^2 + b^2)
= \frac{\pi r^2}{c^2} \cdot c^2 \quad \text{(by Pythagoras’ Theorem)}
= \pi r^2
= \text{Area of the inscribed circle for } \triangle ABC
\]

Hence after any number, say \( n \) steps where, at each step, each sub right-angle triangle is subdivided into two sub right-angle triangles by an altitude, the sum of areas of the inscribed circles in the resulting \( 2^n \) final right-angle triangles is equal to the area of the original inscribed triangle for the original triangle \( \triangle ABC \), i.e.

\[
\text{Area} = \pi \left( \frac{2S}{a+b+c} \right)^2
= \pi \left( \frac{ab}{a+b+c} \right)^2
\]

where the sides are \( a, b, c \) and \( c \) is the length of the hypotenuse. \( \square \)

**Problem 6**
Show how to cut a square of side length 1 by straight lines, so that the resulting pieces can be assembled to form a rectangle in which the ratio of sides is 3 : 1.

\[
\begin{array}{c}
\text{Unit square} \\
\end{array} \quad \overset{\longrightarrow}{\quad} \quad \begin{array}{c}
3:1 \text{ rectangle} \\
\end{array}
\]

**Proof.** [Solution 1] Since we start with a unit square of area 1, and the desired rectangle has sides in ratio 3:1, if the sides are \( x \) and \( 3x \) then

\[ 1 = 3x^2 \quad \Rightarrow \quad x = \frac{1}{\sqrt{3}}. \]

Go up \( \frac{1}{\sqrt{3}} \) on one side of the unit square and draw a line parallel to the other two sides of length \( \sqrt{3} \). Complete the \( \frac{1}{\sqrt{3}} \times \sqrt{3} \) rectangle \( BGJA \) as shown in the diagram.
Now draw the line \( CJ \), the longest line segment from a vertex of the unit square to a vertex of the rectangle.

We claim \( \triangle CBE \equiv \triangle HIJ \) and \( \triangle CDH \equiv \triangle EGJ \), and hence to perform the transformation, we make two straight line cuts in the unit square along \( CH \) and \( BE \), then slide down \( \triangle CDH \) on the line \( CH \) to the position of \( \triangle EGJ \) and move \( \triangle CBE \) to the position of \( \triangle HIJ \).

We could verify the claim in various ways — here we use coordinate geometry. Let \( AC \) be the positive \( y \)-axis and \( AJ \) the positive \( x \)-axis with \( A \) the origin. Hence we have coordinates:

\[
A(0,0), \; C(0,1), \; D(1,1), \; I(1,0) \]

\[
B \left( 0, \frac{1}{\sqrt{3}} \right), \; F \left( 1, \frac{1}{\sqrt{3}} \right), \; G \left( \sqrt{3}, \frac{1}{\sqrt{3}} \right), \; J(\sqrt{3},0)
\]

Line \( CJ \) has equation

\[
y - 0 = \frac{1 - 0}{0 - \sqrt{3}}(x - \sqrt{3}) \iff y = -\frac{1}{\sqrt{3}}(x - \sqrt{3}).
\]

Hence at \( E \), \( y = 1/\sqrt{3} \) and so

\[
\frac{1}{\sqrt{3}} = -\frac{x}{\sqrt{3}} + 1 \iff x = \sqrt{3} - 1 \implies E \left( \sqrt{3} - 1, \frac{1}{\sqrt{3}} \right)
\]

and at \( H \), \( x = 1 \) and so

\[
y = -\frac{1}{\sqrt{3}}(1 - \sqrt{3}) = 1 - \frac{1}{\sqrt{3}} \implies H \left( 1, 1 - \frac{1}{\sqrt{3}} \right).
\]

Hence \( \triangle CBE \equiv \triangle HIJ \) as

1. \( \angle CBE = \angle HIJ \) = a right angle.

2. \( CB = HI = 1 - \frac{1}{\sqrt{3}} \).
3. \(BE = JI = \sqrt{3} - 1\).

And \(\triangle CDH \equiv \triangle EGJ\) as

1. \(\angle CDH = \angle EGJ = \) a right angle.

2. \(CD = EG = 1\).

3. \(DH = GJ = \frac{1}{\sqrt{3}}\).

\(\square\)

**Proof.** [Solution 2] This solution is essentially the same as a general two-cut construction to convert a rectangle with sides \(x\) and 1 into a square where \(1 \leq x \leq 4\), as in Joseph S. Madachy, *Madachy’s Mathematical Recreations* (Dover Publications, 1979) on page 12 in Chapter 1 Geometric Dissections.

\[
\begin{align*}
AD &= x, \ CD = 1, \ ED = b, \ EF = a, \ GC = c
\end{align*}
\]

Make two straight line cuts, \(GD\) and \(EF\), then slide up \(\triangle GCD\) up and left along line \(GD\), and move \(\triangle FED\) up and left into position of \(\triangle GE_2D\) in the square.

For consistency of slope of line \(GD\) in the rectangle (or \(\triangle GCD \parallel \triangle DEF\)), we must have

\[
\frac{a}{b} = \frac{1}{c}
\]
and to obtain a square we must have

\[ c = x - b = 1 + a. \]

Therefore,

\[
\begin{align*}
  b &= a + a^2 \quad \text{and} \quad x = 1 + a + b \\
  &= 1 + a + a + a^2 \\
  &= 1 + 2a + a^2 \\
  &= (1 + a)^2
\end{align*}
\]

so

\[ a = \sqrt{x} - 1 \]

and so we need

\[
\begin{align*}
  x &\geq 1 \quad \text{and} \quad b = a + a^2 = (\sqrt{x} - 1) + x - 2\sqrt{x} + 1 \\
  &= x - \sqrt{x}, \\
  c &= 1 + a = \sqrt{x}.
\end{align*}
\]

This works provided also that

\[ c \leq x \quad \text{and} \quad a \leq 1. \]

Now,

\[
\begin{align*}
  c &= \sqrt{x} \leq x \quad \text{iff} \quad x \geq 1 \quad \text{and} \quad a = \sqrt{x} - 1 \leq 1 \quad \text{iff} \quad \sqrt{x} \leq 2 \quad \text{or} \quad x \leq 4.
\end{align*}
\]

\[ \square \]
Problem 1
The integer part of the real number $x$, written $[x]$, is the unique integer $m$, such that

$$m \leq x < m + 1.$$  

For example,

$$\left[ 3 + \frac{1}{2} \right] = 3 \text{ and } \left[ -3 - \frac{1}{2} \right] = -4.$$ 

Let $k$ and $n$ be positive integers. Evaluate the expression

$$\left[ \frac{n}{k} \right] + \left[ \frac{n+1}{k} \right] + \cdots + \left[ \frac{n+k-1}{k} \right].$$

Proof. [Solution] We set

$$A_n = \left[ \frac{n}{k} \right] + \left[ \frac{n+1}{k} \right] + \cdots + \left[ \frac{n+k-1}{k} \right].$$

Now, we note that

$$A_0 = \left[ 0 \right] + \left[ \frac{1}{k} \right] + \cdots + \left[ \frac{k-1}{k} \right] = 0 + 0 + \cdots + 0 = 0.$$ 

Also, we note that

$$A_{n+1} = A_n - \left[ \frac{n}{k} \right] + \left[ \frac{n}{k} + 1 \right] = A_n + 1.$$ 

Consequently,

$$A_n = n.$$ 

Problem 2
Players $A$ and $B$ play the following game:

1. the game starts with 1000 counters;

2. at every move, a player subtracts $n$ counters, where $n$ is some power of 2, including $2^0 = 1$;

3. the player cannot subtract more counters than are present at any given stage;

4. the player who first reaches 0 is the winner.
Find the optimal strategy and the winner, if player $A$ starts the game.

**Proof.** [Solution] Player $A$ always wins by adhering to the strategy in which the number of counters available before every move of player $B$ is to be a multiple of 3:

1. at the start of the game, the number of counters, $P$, is

   $$ P \equiv 1 \mod 3; $$

2. if the number of counters before each move of player $B$ is a multiple of 3, then the number of counters before the following move of player $A$ is either

   $$ P \equiv 1 \mod 3 \text{ or } P \equiv 2 \mod 3; $$

3. by subtracting either

   $$ 2^0 = 1 \text{ or } 2^1 = 2, $$

player $A$ makes sure that the number of counters available to player $B$ on his next move is again a multiple of 3.

\[ \square \]

**Problem 3**

Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + \cdots + a_n = 1$. Prove that

$$ \frac{1}{a_1} + \cdots + \frac{1}{a_n} \geq n^2. $$

**Proof.** [Solution] By the inequality between arithmetic and geometric mean,

$$ \frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{\frac{1}{n}} \quad \text{and} \quad \frac{1}{n} \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \geq \left( \frac{1}{a_1 a_2 \cdots a_n} \right)^{\frac{1}{n}}. $$

Multiplying together,

$$ (a_1 + a_2 + \cdots + a_n) \times \left( \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \geq n^2. $$

That is,

$$ \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq n^2, $$

given

$$ a_1 + a_2 + \cdots + a_n = 1. $$

\[ \square \]

**Problem 4**

Given two circles of radius 1 with their centres one unit apart, a point $A$ is chosen on the first circle. Two other points $B_1$ and $B_2$ are chosen on the second circle, so that they are symmetric with respect to the line connecting the centres of the circles. Prove that

$$ (AB_1)^2 + (AB_2)^2 \geq 2. $$
Proof. [Solution] By cosine theorem,

\[ AB_1^2 = MA^2 + B_1M^2 - 2MA \times B_1M \times \cos \alpha \quad \text{and} \quad AB_2^2 = MA^2 + B_2M^2 + 2MA \times B_2M \times \cos \alpha. \]

Adding together

\[ AB_1^2 + AB_2^2 = 2 \times (MA^2 + B_1M^2). \]

On the other hand, the distance

\[ MA \]

is minimised if

\[ A = A_{\text{min}}, \]

so by Pythagoras’ Theorem

\[ MA_{\text{min}}^2 + MB_1^2 = 1. \]

\[ \square \]

Problem 5
Let \( n \) be a positive integer.

a) Explain why the set \( S = \{1, 2, \ldots, n\} \) can be partitioned into two non-empty disjoint subsets in exactly \( 2^{n-1} - 1 \) ways.
b) Find the number of ways the set \{1, 2, \ldots, 7\} can be partitioned into three non-empty disjoint subsets.

\textbf{Proof.} [Solution] a) There are \(2^n\) subsets of the set

\[ S = \{1, 2, \ldots, n\}. \]

If \(A\) is one of these, then

\[ A, S \setminus A \]

gives a partition. Since order is unimportant, there are \(2^{n-1}\) such partitions. Since, the partition

\[ \emptyset, S \]

is also included in the above computation, the total number of non-empty partitions is \(2^{n-1} - 1\).

b) Let \(a_n\) be the number of partitions of set

\[ S_n = \{1, 2, \ldots, n\} \]

into three non-empty subsets. We see a recurrence relation for the sequence

\[ \{a_n\}_{n=1}^{\infty} \]

as follows: we can partition the set

\[ S_{n-1} = \{1, 2, \ldots, n - 1\} \]

in \(a_{n-1}\) ways and there are 3 subsets in which to place the number \(n\). Also, we could partition \(S_n\) as

\[ A, B, \{n\}, \]

where

\[ A \cup B = S_{n-1} \text{ and } A \neq \emptyset, B \neq \emptyset. \]

By part a), we have exactly

\[ 2^{n-1} - 1 \]

possibilities for the latter. So, we arrive at

\[ a_1 = a_2 = 0, \quad a_3 = 1, \quad a_n = 3a_{n-1} + 2^{n-2} - 1, \quad n \geq 4. \]

Hence,

\[ a_4 = 3a_3 + 4 - 1 = 6, \]
\[ a_5 = 3a_4 + 8 - 1 = 25, \]
\[ a_6 = 3a_5 + 16 - 1 = 90, \]
\[ a_7 = 3a_6 + 31 - 1 = 301. \]

\hfill \square

\textbf{Problem 6}

Show how to cut a square of side length 1 by straight lines, so that the resulting pieces can be assembled to form a rectangle in which the ratio of sides is \(3 : 1\).
Proof. [Solution] See solution in Junior section.