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Solutions 1451–1460

Q1451 Use the ideas of the solution to problem 1443 (previous issue) to find *without calculus* the maximum value of

$$\frac{x}{(x^2+a^2)^2}\ ,$$

where a is a positive real number.

SOLUTION The maximum (if it exists) is clearly a positive number; for convenience we call it $1/c$. Therefore we want

$$\frac{x}{(x^2+a^2)^2} \le \frac{1}{c}$$

for all x, with equality holding for at least one value of x. Multiplying out this inequality gives

$$x^4 + 2a^2x^2 - cx + a^4 \ge 0\ .$$

Now if equality holds for some x, say $x = \alpha$, the left-hand side must have a factor $x-\alpha$; but if the left-hand side is never negative, then $x = \alpha$ must in fact be a double root: that is, $(x-\alpha)^2$ is a factor. Dividing out to obtain a quotient and remainder,

$$\begin{aligned} x^4 &+ 2a^2x^2 - cx + a^4 \\ &= (x-\alpha)^2(x^2 + 2\alpha x + 3\alpha^2 + 2a^2) \\ &\quad + (4\alpha^3 + 4a^2\alpha - c)x - (3\alpha^4 + 2a^2\alpha^2 - a^4)\ , \end{aligned}$$

and we require the remainder to be zero. Thus

$$3\alpha^4 + 2a^2\alpha^2 - a^4 = 0 \tag{1}$$

and

$$4\alpha^3 + 4a^2\alpha - c = 0\ . \tag{2}$$

Now take 3α times equation (2) minus 4 times (1):

$$4a^2\alpha^2 - 3\alpha c + 4a^4 = 0\ , \tag{3}$$

and now a^2 times (2) minus α times (3):

$$3\alpha^2 c - a^2 c = 0\ .$$

Since c cannot be zero we have $\alpha^2 = a^2/3$, giving the location of the double root, and then equation (2) yields

$$c = 4\alpha^3 + 4a^2\alpha = \pm\frac{16a^3}{3\sqrt{3}}\ .$$

As c is positive we reject the negative root, and so the required maximum is

$$\frac{1}{c} = \frac{3\sqrt{3}}{16a^3}\ .$$

Alternative solution Instead of investigating the whole problem from the beginning, we can reduce it to the special case we saw in problem 1443 (solution in volume 50 number 2). If we take $y = \sqrt{3}\,x/a$, then the expression of interest is

$$\frac{x}{(x^2+a^2)^2} = \frac{ay/\sqrt{3}}{((a^2y^2/3)+a^2)^2} = \frac{3\sqrt{3}}{a^3}\,\frac{y}{(y^2+3)^2}\ .$$

Since we already know that the maximum value of $y/(y^2+3)^2$ is $1/16$, the answer to the present problem is

$$\frac{3\sqrt{3}}{16a^3}$$

as above.

Q1452 As in problem 1442 (see the statement and solution in the previous issue), a particle is projected at a $45°$ angle from one corner of a 2014×1729 rectangle. Find the first occasion on which the particle hits the top wall and then the bottom, or the bottom and then the top, without hitting the left or right wall in between.

SOLUTION Suppose that the particle hits the top or bottom wall for the mth time, and then hits the bottom or top wall next, without hitting the left or right wall in between. If this is the first time this has happened, then it has already hit the left and right walls $m-1$ times. The total distance travelled vertically is $1729m$; the distance travelled horizontally is $2014(m-1)$ plus a bit more. Let the "bit more" be r; then $1729m = 2014(m-1)+r$, and if after the mth impact on the top or bottom wall the particle is to hit the bottom or top wall next, we need $r + 1729 < 2014$. Simplifying all these conditions, we want

$$2014 = 285m + r \quad \text{with} \quad 0 < r < 285\ ;$$

in other words, we want to find the quotient m when 2014 is divided by 285. This gives $m = 7$; therefore the particle does not hit the left or right wall in between the 7th and 8th impacts on the top and bottom walls.

NOW TRY problem 1461.

Q1453 In the town of truth–tellers and liars from problem 1444 (see the statement and solution in the previous issue), I meet four more people. I ask each of them, "How many of **the other three** are liars?"

George says "One". Helen says, "I don't know." Ian says, "Three." Jacqui says, "Two". Are these people truth–tellers or liars?

SOLUTION As in the solution to the previous problem, Helen must be a truth–teller (liars don't admit their ignorance).

Now consider Ian's answer. He has heard Helen's reply and knows that she is a truth–teller. Thus, Ian has given a false answer, and he must be a liar.

Suppose that Jacqui is a liar. She has heard the others' statements and she can work out, just as we have done, that Helen is a truth–teller and Ian is a liar. So she knows for sure that "None" and "Three" are false answers. Therefore, as she has given the answer "Two", she must know for sure that that is false too. That is, she knows that George is a truth–teller. But in that case George has given a false answer (the correct answer is "Two"), which is impossible: truth–tellers either give the right answer or say "I don't know." Therefore, the assumption that Jacqui is a liar is untenable, and she must be a truth–teller. This being the case, and since she knows that Helen is a truth–teller, she must also know that George is a liar. (She could not deduce this from what he said, so she must have known it already.)

So, George and Ian are liars, while Helen and Jacqui are truth–tellers.

Comment. Notice that George, although he is a liar, has given a true answer! It must be that he had no idea at all whether the others were truth–tellers or not, so he gave a random answer which by accident happened to be correct.

Q1454 If the roots of the equation $x^3 + x + 2014 = 0$ are $\tan\alpha,\ \tan\beta$ and $\tan\gamma$, can you evaluate $\tan(\alpha+\beta+\gamma)$?

SOLUTION Relating the roots and coefficients of the polynomial, we have

$$\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha = 1 \ ,$$

which can be written

$$\tan\gamma(\tan\alpha + \tan\beta) = 1 - \tan\alpha\tan\beta \ .$$

Now $\tan\gamma \neq 0$, because 0 is not a root of the given cubic; furthermore $\tan\alpha + \tan\beta \neq 0$, for otherwise we have

$$\tan\alpha + \tan\beta = 0 \ \Rightarrow\ 1 - \tan\alpha\tan\beta = 0 \ \Rightarrow\ 1 + \tan^2\alpha = 0$$

which is impossible. Hence $1 - \tan\alpha\tan\beta \neq 0$ and we have

$$\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{1}{\tan\gamma} \ .$$

Using the tan–of–a–sum formula,

$$\tan(\alpha+\beta) = \frac{1}{\tan\gamma} = \cot\gamma = \tan\Bigl(\frac{\pi}{2} - \gamma\Bigr) \ .$$

But this gives

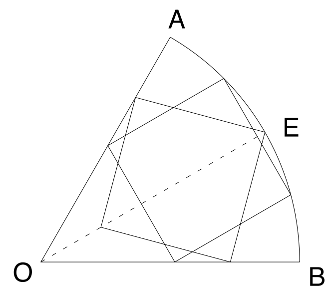
$$\alpha + \beta = \frac{\pi}{2} - \gamma + k\pi$$

for some integer k, and hence

$$\alpha + \beta + \gamma = \Bigl(k + \frac{1}{2}\Bigr)\pi \ ;$$

therefore $\tan(\alpha+\beta+\gamma)$ is undefined and cannot be evaluated.

Q1455 The diagram shows two squares inscribed in a 60° sector of a circle; the points A, B and E are on the circular arc, and each square is symmetric about the angle–bisector OE. Prove that the squares have the same size.



SOLUTION Since OE bisects $\angle O$ and $DC \perp OE$, the triangle $\triangle OCD$ is isosceles; and since it includes a 60° angle, it is equilateral. Therefore $CD = CO$, and $CD = CB$ (sides of the same square); so $CO = CB$, and $\triangle OCB$ is isosceles. Moreover,

$$\angle OCB = \angle OCD + \angle DCB = 60^\circ + 90^\circ = 150^\circ ,$$

and so $\angle CBO = 15^\circ$. Hence

$$\angle OBD = \angle CBD - \angle CBO = 45^\circ - 15^\circ = 30^\circ .$$

Now consider $\triangle OBD$ and $\triangle EOH$. The former has angles $\angle O = 45^\circ$ and $\angle B = 30^\circ$, with included side OB equal to the radius of the circle; and the other has the same properties. Therefore the two triangles are congruent, $HE = DO = DC$, and we are finished.

Q1456 The positive integers a and b have no common factor. The positive integer n is a multiple of both a and b; exactly half the numbers from 1 to n are multiples of a or of b but not both. Find a and b.

SOLUTION The number of integers from 1 to n that are multiples of a is n/a, and the number that are multiples of b is n/b. Since a and b have no common factor, an integer is a multiple of both a and b if and only if it is a multiple of ab. The number of such integers is n/ab; they will be counted among the multiples of a and among the multiples of b, but should not have been counted at all. So the given condition can be written as

$$\frac{n}{a} + \frac{n}{b} - 2\frac{n}{ab} = \frac{n}{2} .$$

Cancelling n and rearranging,

$$ab - 2a - 2b + 4 = 0 ,$$

and the left-hand side can be factorised to give

$$(a - 2)(b - 2) = 0 .$$

Therefore either a or b must be 2, and since they have no common factor the other one must be an odd number.

NOW TRY problem 1463.

Q1457 Mitchell and Dale are playing a game with dice. Mitchell has a die with five sides (each equally likely to show up) and Dale has a normal six–sided die. The two throw their dice alternately, with Mitchell going first. The first to throw a 1 wins. What are the winning chances of the two players?

SOLUTION Let m be Mitchell's probability of winning. There is a $\frac{1}{5}$ chance that he will win on his first throw. For Mitchell to win the game, but not on his first throw, he must fail on his first throw (probability $\frac{4}{5}$), and Dale must fail on his first throw (probability $\frac{5}{6}$). From this point on it is just as if the game is played again from the start, and so Mitchell's winning probability is again m. Putting all this together gives the equation

$$m = \frac{1}{5} + \Bigl(\frac{4}{5}\Bigr)\Bigl(\frac{5}{6}\Bigr)m ,$$

which is easily solved to give $m = \frac{3}{5}$. Dale's winning probability is therefore $d = 1 - m = \frac{2}{5}$, which can be checked by setting up an equation as we did for Mitchell.

NOW TRY problem 1462.

Q1458 Add up the numbers $0, 1, 2, 3, 4, \ldots$ successively to get

$$0\,,\ 1\,,\ 3\,,\ 6\,,\ 10\,,\ 15\,,\ 21\,,\ 28\,, \ldots\ . \qquad (*)$$

The remainders when these numbers are divided by 8 are

$$0\,,\ 1\,,\ 3\,,\ 6\,,\ 2\,,\ 7\,,\ 5\,,\ 4\,, \ldots\ .$$

Notice that every possible remainder from 0 to 7 appears. For which numbers other than 8 is this true? That is: determine all positive integers m such that if we continue the sequence $(*)$ indefinitely and then find the remainder when each term is divided by m, all possible remainders from 0 to $m - 1$ appear.

SOLUTION This will work if and only if m is a power of 2.

First we prove that it does not work if m is odd. In that case the mth term in the sequence is

$$0 + 1 + 2 + \cdots + (m - 1) = \Bigl(\frac{m - 1}{2}\Bigr)m ,$$

which has remainder 0 when divided by m. To continue the process we shall add $m, m+1, m+2, \ldots$; and since we are taking remainders after division by m, this is the same as adding $0, 1, 2, \ldots$. Therefore all the remainders of the sequence from this point on will repeat those we have seen already. That is, the only way we will get all m possible remainders is if we already have m different remainders in the first m terms. But this is not the case, for the mth term in the sequence gives remainder 0, the same as the first term.

Next we show that the procedure does not give all possible remainders if m is not a power of 2. This follows from the previous argument, for if m is not a power of 2 then it has an odd factor $n > 1$; if the sequence contains all remainders after division by m, then it contains all remainders after division by n; but we have just seen that this is not so.

Finally, we show that all remainders are obtained if m is a power of 2, say $m = 2^t$. To do this we prove that no two of the first m terms in the sequence give the same remainder. Suppose that the ath term and the bth term give the same remainder, where $1 \le a < b \le m$. Then m is a factor of the difference

$$\begin{aligned}(0+1+\cdots+(b-1)) - (0+1+&\cdots+(a-1))\\ &= a+(a+1)+\cdots+(b-1)\\ &= \frac{(a+b-1)(b-a)}{2}\ ,\end{aligned}$$

and so 2^{t+1} is a factor of $(a+b-1)(b-a)$. Now the factors $a+b-1$ and $b-a$ cannot both be even (because their sum is odd); and so 2^{t+1} must be a factor of one or the other. But we have

$$0 < a+b-1 < 2m = 2^{t+1} \quad \text{and} \quad 0 < b-a < m < 2^{t+1}\ ,$$

so this is not possible. We have shown that the first m remainders we obtain are all different; so they must include all possible remainders.

Q1459 It can be proved from the Peano axioms for arithmetic (see Michael Deakin's article in *Parabola Incorporating Function* Vol 50, No 1) that every number except 1 is the successor of some number: that is, if $x \neq 1$ then $x = y^+$ for some y. Use this result, together with the axioms and the definitions of addition and multiplication (equations (1), (2), (8) and (9) in Michael's article) to prove the following.

(a) There are no numbers x, y such that $x + y = 1$.

(b) If $x * y = 1$ then $x = 1$ and $y = 1$.

SOLUTION

(a) Suppose that $x + y = 1$. There are two possibilities: $y = 1$ or $y \neq 1$. In the first case, if $x + y = 1$ then $x + 1 = 1$, so by (1) we have $x^+ = 1$, which is impossible as it contradicts axiom (iii). In the second case we can write $y = z^+$; then if $x + y = 1$ we use (2) to see that $(x+z)^+ = x + z^+ = x + y = 1$, which again is a contradiction.

(b) Suppose that $x * y = 1$. If $y \neq 1$ then again we can write $y = z^+$; so $x * z^+ = 1$; and by equation (9) we have $(x * z) + x = 1$; but in view of (a), this is impossible. Therefore we have $y = 1$; and so equation (8) shows that $x = x * 1 = x * y = 1$.

Q1460 One more question about the coin–sharing problem (see the previous two issues for the basic questions): what happens if there are 19 people sharing out 10 coins according to the same rules?

SOLUTION Following through the same reasoning as in previous issues, Richard will propose to give Adam, Betty, …, Quyen and himself $0, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1$ coins respectively. (Please see previous solutions and work out the details for yourself!) Sam knows this and thinks: if my proposal is rejected I get nothing. But to get my proposal accepted I have to give away ten coins (one each to Adam, Ellen, George, Ivan, Kerry, Martha, Olwen and Quyen and two coins to one of the others), so I still get nothing. Therefore all proposals are alike to me, and I'll just choose one of the 13123110 options entirely at random.

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