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A Method to find the Sums of Polynomial Functions at Positive Integer Values

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When learning the intuition behind definite integration, calculus students often learn how to find the area under a curve by using a Riemann sum. Often, when attempting to find the area under polynomial curves by this method, students are limited by how many formulas they know for the sums of monomials of a positive integer degree n.

The most commonly known formula of this variety is for the sum of monomials of degree 1, namely

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}\,.$$

Less common are the formulas for the sum of monomials of degree 2 and 3, namely

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}\,.$$

Most calculus students are taught to memorize these formulas, and are thus not able to find the sums of polynomials of degrees higher than 3. Further research into the area yields Faulhaber's formula, which involves more complex concepts such as the Bernoulli numbers, with which students are often unfamiliar. In this paper, I show a method for deriving these summations for polynomials of higher degrees without using these complex concepts.

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1 Method

For some fixed integer $p \geq 1$, let $S_p(n)$ be a function on the positive integers n such that

$$S_p(n) = \sum_{i=0}^{n} i^p .$$

The function $S_p(n)$ can be recursively defined as follows:

$$\begin{aligned} &S_p(0) = 0 \\ &S_p(n) = n^p + S_p(n-1) \qquad (n \geq 1) . \end{aligned}$$

This can be rewritten as

$$\begin{aligned} &S_p(0) = 0 \\ &S_p(n) - S_p(n-1) = n^p \qquad (n \geq 1) . \end{aligned}$$

If $S_p(n)$ can be written as a polynomial of degree t, then so too can $S_p(n-1)$. For each integer $m = 0, \ldots, t$, let C_m be the coefficient of n^m in the polynomial representation of $S_p(n)$. Since $S_p(n)$ is of degree t, it follows that $C_t \neq 0$. This means that

$$\begin{aligned} S_p(n) &= C_t n^t + C_{t-1} n^{t-1} + \cdots + C_1 n + C_0 \qquad\qquad (C_t \neq 0) \\ S_p(n-1) &= C_t (n-1)^t + C_{t-1}(n-1)^{t-1} + \cdots + C_1(n-1) + C_0 \\ &= (C_t n^t - tC_t n^{t-1} + \tbinom{t}{2} C_t n^{t-2} - \cdots) + \\ &\quad (C_{t-1} n^{t-1} - (t-1)C_{t-1} n^{t-2} + \cdots) + \\ &\quad (C_{t-2} n^{t-2} - \cdots) + \cdots \end{aligned}$$

With this notation, we can write out part of the standard polynomial representation of $S_p(n) - S_p(n-1)$ as

$$\begin{aligned} S_p(n) - S_p(n-1) &= (C_t - C_t) n^t + (C_{t-1} - C_{t-1} + tC_t) n^{t-1} + \\ &\quad (C_{t-2} - C_{t-2} + (t-1)C_{t-1} - \tbinom{t}{2} C_t) n^{t-2} + \cdots \\ &= tC_t n^{t-1} + ((t-1)C_{t-1} - \tbinom{t}{2} C_t) n^{t-2} + \cdots \end{aligned}$$

As is evident from this simplification, the leading term of the polynomial $S_p(n)$ cancels out with the leading term of the polynomial $S_p(n-1)$, leaving a polynomial of degree $t-1$. Since $C_t \neq 0$, the leading coefficient tC_t is non-zero, so the degree of the resulting polynomial must be $t-1$.

Since $S_p(n) - S_p(n-1)$ equals n^p, we know that this polynomial is of degree p. Thus, $t - 1 = p$, so $S_p(n)$ and $S_p(n-1)$ must both be of degree $p+1$. Equivalent polynomials must have equal coefficients, so we can create a system of equations by equating the coefficients of $S_p(n) - S_p(n-1)$ with the coefficients of n^p. Then, we can solve for the coefficients of $S_p(n)$ to find its polynomial representation.

We also must remember that since $S_p(0) = 0$, the constant term of the polynomial representation of $S_p(n)$ is $C_0 = 0$. This means that we can exclude it from all calculations with the understanding that there is no (non-zero) constant term in the polynomial in question.

2 Example: Deriving the Sum of a Quartic Monomial

Problem: Find the function $S_4(n) = \sum_{i=0}^{n} i^4$.

We start by writing out the recursive definition of $S_4(n)$ and expanding, with the knowledge that $S_4(n)$ can be written as a 5^{th} degree polynomial:

$$\begin{aligned}
n^4 &= S_4(n) - S_4(n-1) \\
&= (C_5n^5 + C_4n^4 + C_3n^3 + C_2n^2 + C_1n) - \\
&\quad [C_5(n-1)^5 + C_4(n-1)^4 + C_3(n-1)^3 + C_2(n-1)^2 + C_1(n-1)] \\
&= (C_5n^5 + C_4n^4 + C_3n^3 + C_2n^2 + C_1n) - \\
&\quad [C_5(n^5 - 5n^4 + 10n^3 - 10n^2 + 5n - 1) + \\
&\quad C_4(n^4 - 4n^3 + 6n^2 - 4n + 1) + \\
&\quad C_3(n^3 - 3n^2 + 3n - 1) + \\
&\quad C_2(n^2 - 2n + 1) + \\
&\quad C_1(n-1)]
\end{aligned}$$

Next, we can rewrite the polynomial in standard form:

$$\begin{aligned}
n^4 &= C_5(5n^4 - 10n^3 + 10n^2 - 5n + 1) + \\
&\quad C_4(4n^3 - 6n^2 + 4n - 1) + \\
&\quad C_3(3n^2 - 3n + 1) + \\
&\quad C_2(2n - 1) + \\
&\quad C_1(1) \\
&= (5C_5)n^4 \\
&\quad (-10C_5 + 4C_4)n^3 + \\
&\quad (10C_5 + -6C_4 + 3C_3)n^2 + \\
&\quad (-5C_5 + 4C_4 - 3C_3 + 2C_2)n + \\
&\quad (C_5 - C_4 + C_3 - C_2 + C_1)
\end{aligned}$$

Since we know that this polynomial is equivalent to n^4 for all values of n, we can equate the coefficients of these polynomials, yielding the following system of equations:

$$\begin{aligned}
5C_5 &= 1 \\
-10C_5 + 4C_4 &= 0 \\
10C_5 - 6C_4 + 3C_3 &= 0 \\
-5C_5 + 4C_4 - 3C_3 + 2C_2 &= 0 \\
C_5 - C_4 + C_3 - C_2 + C_1 &= 0
\end{aligned}$$

Solving these equations yields the resulting values for each variable:

$$C_5 = \frac{1}{5} \qquad C_4 = \frac{1}{2} \qquad C_3 = \frac{1}{3} \qquad C_2 = 0 \qquad C_1 = -\frac{1}{30}\,.$$

This means that

$$S_4(n) = \sum_{i=0}^{n} i^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n = \boxed{\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}}.$$

3 Conclusion

In summary, by leveraging certain properties of polynomials, it is possible to derive the sums of polynomials evaluated at positive integer values without any further knowledge than what is taught in a standard pre-calculus class and without memorizing any formulas. The question of summing polynomials with fractional powers needs to be explored further. In addition, for larger values of p, the system of equations necessary to solve these problems could potentially be expressed more easily using matrix algebra.

This method can be helpful in evaluating many mathematical expressions such as the integral

$$\int_0^s \lfloor x \rfloor x^n dx$$

for a non-negative integer n. This method also makes calculating the formulas of many figurate numbers very straightforward.