The Enclosure

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An enclosure of length 1 unit is constructed around two adjoining walls of unlimited length. It is made of \( n \geq 2 \) straight sections, referred to as an \( n \)-enclosure, designed so as to maximise the enclosed area \( A_n(\omega) \), where \( \omega \leq \pi \) is the angle formed by the walls.

For \( n = 2 \), the enclosure’s perimeter equals 1 if each wall has length

\[
x = \frac{1}{2} \csc \left( \frac{\omega}{2} \right) \sin \left( \phi + \frac{\omega}{2} \right).
\]

Figure 1: The 2-enclosure.

In terms of the independent variable \( \phi \),

\[
A_2(\omega) = \frac{1}{4} \csc^2 \left( \frac{\omega}{2} \right) \sin \phi \sin \left( \phi + \frac{\omega}{2} \right),
\]

so

\[
\frac{dA_2(\omega)}{d\phi} = \frac{1}{4} \csc^2 \left( \frac{\omega}{2} \right) \sin \left( 2\phi + \frac{\omega}{2} \right) = 0.
\]

The solution to this equation is \( \phi = \frac{\pi}{2} - \frac{\omega}{4} \). The maximum enclosed area is

\[
A_2^*(\omega) = A_2(\omega) : \phi = \frac{\pi}{2} - \frac{\omega}{4} = \frac{1}{8} \cot \left( \frac{\omega}{4} \right).
\]

Over the other side of the walls, \( \bar{x} \) and \( A_2(\bar{\omega}) \) where

\[
\bar{\omega} := 2\pi - \omega
\]

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bear the same expressions as $x$ and $A_2(\omega)$ with $\phi, \omega$ substituted by $\bar{\phi}, \bar{\omega}$. Hence,

$$A_2^*(\bar{\omega}) = \frac{1}{8} \cot \left( \frac{\bar{\omega}}{4} \right) = \frac{1}{8} \tan \left( \frac{\omega}{4} \right).$$

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Figure 2: The 3-enclosure.

In terms of the independent variables $y, \nu$

$$A_3(\omega) = \frac{1}{2} y^2 \sin(2\nu) + y(1 - 2y) \sin \nu + \cot \left( \frac{\omega}{2} \right) \left( \frac{1}{2} - y(1 - \cos \nu) \right)^2,$$

so

$$\frac{\partial A_3(\omega)}{\partial y} = y \sin(2\nu) + (1 - 4y) \sin \nu - 2 \cot \left( \frac{\omega}{2} \right) (1 - \cos \nu) \left( \frac{1}{2} - y(1 - \cos \nu) \right) = 0.$$

Hence,

$$\frac{1}{y} \frac{\partial A_3(\omega)}{\partial \nu} = y \cos(2\nu) + (1 - 2y) \cos \nu - 2 \cot \left( \frac{\omega}{2} \right) \cos \nu \left( \frac{1}{2} - y(1 - \cos \nu) \right) = 0.$$

where $\partial$ denotes the partial differentiation.

Multiplying the former by $\sin \nu$, the latter by $1 - \cos \nu$ and subtracting one expression from the other yields $y = \frac{1}{3}$. Substituting this value into either expression leads to $\nu = \frac{\omega}{3}$. The maximum enclosed area is

$$A_3^*(\omega) = A_3(\omega) : y \mapsto \frac{1}{3}, \nu \mapsto \frac{\omega}{3} = \frac{1}{12} \cot \left( \frac{\omega}{6} \right).$$

Over the other side of the walls, $\bar{x}, \bar{y}$ and $A_3(\bar{\omega})$ are as per $x, y$ and $A_3(\omega)$ with $\nu, \omega$ substituted by $\bar{\nu}, \bar{\omega}$. Hence,

$$A_3^*(\bar{\omega}) = \frac{1}{12} \cot \left( \frac{\bar{\omega}}{6} \right) = \frac{1}{12} \cot \left( \frac{\pi}{3} - \frac{\omega}{6} \right).$$
Recapping this: For either case, the maximum enclosed area is a tessellation by isosceles triangles: 2 triangles of equal sides equal to $\frac{1}{4} \csc \left( \frac{\omega}{4} \right)$ for $n = 2$, and 3 triangles of side lengths each equal to $\frac{1}{6} \csc \left( \frac{\omega}{6} \right)$ for $n = 3$.

The common apex of these triangles will subsequently be referred to as the centre of the enclosure. Staying in that vein spawns the following proposition:

“An $n$-enclosure holds the maximum area if its centre of tessellation, by $n$ isosceles triangles, coincides with the junction of the walls.”

The tessellating triangles have base equal to $\frac{1}{n}$ and apex angles $\frac{\omega}{n}$; hence, the maximum area is

$$A^*_n(\omega) = \frac{1}{4n} \cot \left( \frac{\omega}{2n} \right).$$

By analogy, the area of a regular $n$-gon of perimeter 1 is

$$H_n = \frac{1}{4n} \cot \left( \frac{\pi}{n} \right).$$

Thus,

$$A^*_n \left( \frac{2p\pi}{q} \right) = \frac{q}{p} H_{\frac{2q}{2p}}; \quad (p, q) = 1, p \leq \frac{q}{2}$$

which is the $\frac{p}{q}$-th fraction of a regular $\frac{qn}{p}$-gon of perimeter $\frac{q}{p}$ for $n = p, 2p, 3p, \ldots$.

For example, $A^*_5(2) \div A^*_5 \left( \frac{5\pi}{8} \right) = \frac{8}{3} H_8$ or $A^*_7(2) \div A^*_7 \left( \frac{7\pi}{11} \right) = \frac{22}{7} H_{22}$.

With $\omega + \bar{\omega} = 2\pi$, we have the following inequalities

- $A^*_n(\bar{\omega}) \leq 2H_{2n} \leq A^*_n(\omega)$,
- $(A^*_n(\omega) A^*_n(\bar{\omega}))^{\frac{1}{2}} = \frac{1}{4n} \left( \cot \left( \frac{\omega}{2n} \right) \cot \left( \frac{\bar{\omega}}{2n} \right) \right)^{\frac{1}{2}} \geq \frac{1}{4n} \cot \left( \frac{\pi}{2n} \right) = 2H_{2n}$.

These imply that $\frac{A^*_n(\omega) + A^*_n(\bar{\omega})}{2} \geq 2H_{2n}$.

The inequality is reversed with the harmonic means of $A^*_n(\omega), A^*_n(\bar{\omega})$, namely

$$\left( \frac{1}{A^*_n(\omega)} + \frac{1}{A^*_n(\bar{\omega})} \right)^{-1} = \frac{1}{2n} \csc \left( \frac{\omega}{2n} \right) \cos \left( \frac{\omega}{2n} \right) \cos \left( \frac{\bar{\omega}}{2n} \right) \leq \frac{1}{2n} \csc \left( \frac{\pi}{n} \right) \cos^2 \left( \frac{\pi}{2n} \right) = \frac{1}{4n} \cot \left( \frac{\pi}{2n} \right) = 2H_{2n},$$

This is rather interesting!

This article concludes with a shift of the enclosure’s centre along the bisector of $\omega$, giving it more “roundness” and visual appeal at the expense of some area.

The enclosure’s perimeter is 1 if each wall has length

$$x = \frac{1}{2n} \csc \left( \frac{\omega}{2} \right) \sin \left( \frac{\theta}{2} \right) \csc \left( \frac{\theta}{2n} \right)$$

so

$$z = \frac{1}{2n} \csc \left( \frac{\omega}{2} \right) \sin \left( \frac{\theta - \omega}{2} \right) \csc \left( \frac{\theta}{2n} \right),$$
Figure 3: Shift of the enclosure’s centre along the bisector of $\omega$.

\[
A_n(\omega; \theta) = \frac{1}{4n} \cot \left( \frac{\theta}{2n} \right) + \frac{1}{4n^2} \sin^2 \left( \frac{\theta}{2} \right) \csc^2 \left( \frac{\omega}{2n} \right) \left( \cot \left( \frac{\omega}{2n} \right) - \cot \left( \frac{\theta}{2n} \right) \right)
\]

\[
\leq A_n^*(\omega)
\]

and

\[
A_n^*(\omega) - A_n(\omega; \theta) = \frac{1}{4n} \sin \left( \frac{\omega}{2n} \right) \left( \cot \left( \frac{\omega}{2n} \right) - \cot \left( \frac{\theta}{2n} \right) \right)
\]

\[
\times \left( \frac{\sin \left( \frac{\omega}{2} \right)}{\sin \left( \frac{\omega}{2n} \right)} - \frac{1}{n} \frac{\sin \left( \frac{\theta}{2} \right) \sin \left( \frac{\theta - \omega}{2n} \right)}{\sin \left( \frac{\theta - \omega}{2n} \right)} \right)
\]

Over the other side of the walls, $(\bar{x}, \bar{z}) = (x, z) : (\omega, \theta) \mapsto (\bar{\omega}, \bar{\theta})$. Thus if $\theta + \omega \geq 2\pi$, then

\[
A_n(\omega; \theta) + A_n(\bar{\omega}; \bar{\theta}) = \frac{1}{2n} \cot \left( \frac{\theta}{2n} \right) - \frac{1}{4n^2} \sin \theta \csc^2 \left( \frac{\theta}{2n} \right)
\]

which is independent of $\omega$.

The special case

\[
\hat{\theta} = \frac{n(\omega + \pi)}{n + 1}
\]

has some appeal, as $AC$ bisects $\angle OAB$. For this choice of $\theta$ we have

\[
A_n(\omega; \hat{\theta}) = \frac{H_{2x}}{4n} \cot \left( \frac{\hat{\theta}}{2n} \right).
\]

For example, $A_3 \left( \frac{\pi}{2}, \frac{9\pi}{8} \right) = \frac{1}{12} + \frac{1}{8} \sqrt{4 + 2\sqrt{2}}$.

Sketch this enclosure with ruler and compass!