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Problems 1481–1490

Q1481 Prove that if the denominator q of a fraction p/q is the number consisting of n digits, all equal to 9, and if p is less than q, then p/q can be written as a repeating decimal in which the repeating part has length n and contains the digits of p, preceded by a sufficient number of 0s to give that length.

Q1482 If the acute angles α and β satisfy the equation

$$(1 - \cot\alpha)(1 - \cot\beta) = 2\,,$$

find the value of $\alpha + \beta$.

Q1483 We have a geometric sequence $a_1, a_2, a_3, \ldots$ for which it is known that

$$a_1 + a_3 + a_5 + a_7 = 5102 \quad \text{and} \quad a_2 + a_4 + a_6 = 2015\,.$$

Evaluate

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2\,.$$

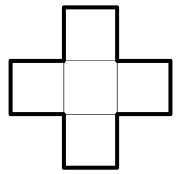
Q1484 Let $n = 6$ and take the numbers in the nth row of Pascal's triangle, leaving out the last of them:

$$1,\ 6,\ 15,\ 20,\ 15,\ 6\,.$$

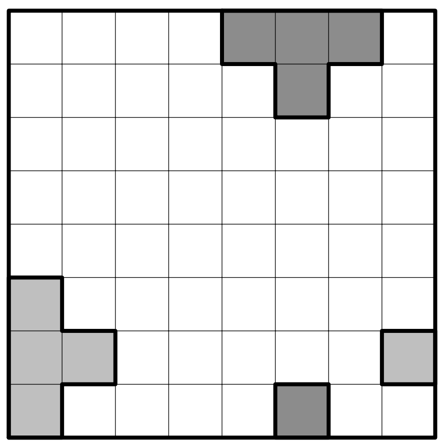
Notice that in this list the first number is a multiple of 1, the second is a multiple of 2, the third is a multiple of 3 and so on, all the way through to the nth number, which is a multiple of n. Prove that this works whenever $n + 1$ is prime.

Q1485 How many 10-digit numbers x are there such that x ends with the digits 2015 and x^2 begins with the digits 2015?

Q1486 In Problem 1479 (solution later this issue) we showed that a maximum of eight crosses like this one



can be placed without overlapping on an 8×8 chessboard. Next, imagine that the top of the board is joined to the bottom and the left side is joined to the right, so that crosses may be placed (for example) as shown, in addition to the "ordinary" placements.



What is now the maximum number of crosses that can be placed on the board?

Q1487 Determine how many values of x satisfy the conditions

$$x^2 - x\lfloor x \rfloor = 20{\cdot}15 , \qquad x \le 2015 .$$

Here the notation $\lfloor x \rfloor$ denotes x rounded to the nearest integer downwards, for example, $\lfloor \pi \rfloor = 3$.

Q1488 Let m be a integer, $m \ge 2$. Prove that there is a cubic polynomial

$$p(x) = x^3 + ax^2 + bx + c$$

with integer coefficients, such that when x is an integer, $p(x)$ is never a multiple of m.

Q1489 The point (s, t) is the centre of a square. Three vertices of the square lie on the parabola $y = x^2$. If $s = \frac{3}{2}$, find the coordinates of all four vertices of the square.

Q1490 Let

$$S = \cos 72^\circ + \cos 144^\circ \quad \text{and} \quad T = \cos 72^\circ - \cos 144^\circ .$$

- Prove that $2ST = -T$.
- Hence find the exact value of $\cos 72^\circ$.