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Lassie and the Loch Ness Monster

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Choose any function $f(n)$ which is defined for all positive integers $n = 1, 2, 3, \ldots$; starting at any point on a sheet of paper, draw a line interval one unit long at an angle $f(1)$ anticlockwise from the horizontal. The angle may be measured in degrees or radians, as you wish, but perhaps the best way is for it to be measured in full circles. Thus, for example, $f(1) = \frac{1}{4}$ corresponds to an interval drawn vertically upwards while $f(1) = -\frac{1}{4}$ will give an interval leading vertically downwards. From the end of the first segment draw a second, again one unit long and this time inclined at an angle $f(2)$ to the horizontal; from the end of this a third, inclined at $f(3)$, and so on. Continuing in this way we obtain a chain of line segments starting at the origin and leading out into the plane.

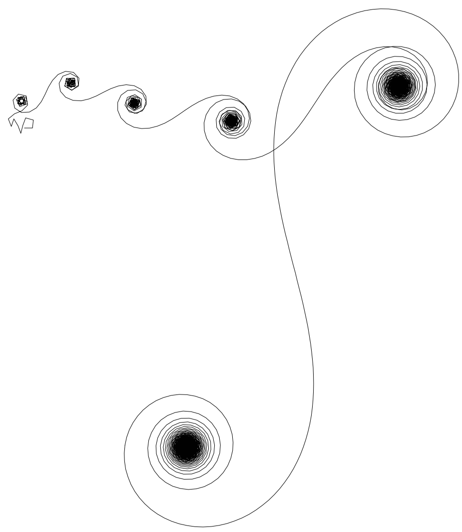

Figure 1: 5000 intervals, $f(n) = (\ln n)^4$

The idea – at any rate, one of the ideas – behind all this is that if we are lucky (or clever!) we may end up with an interesting pattern of line intervals. Of course, plotting all these angles accurately is very difficult, and a much better idea is to make a computer do the work for us. To achieve this we'll need to calculate the (x, y) coordinates of the endpoints of all our intervals, plot them in the plane and join them by straight lines. Let (x_0, y_0) be our starting point, the origin, and let (x_n, y_n) be the far end of the nth interval. Since every interval has length 1 and the angle between the horizontal

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and the nth interval is $2\pi f(n)$ radians, it is easily seen (exercise: draw a diagram to see it!) that

$$x_n = x_{n-1} + \cos\bigl(2\pi f(n)\bigr)\ , \quad y_n = y_{n-1} + \sin\bigl(2\pi f(n)\bigr)\ .$$

Having chosen our function $f(n)$, therefore, we can start with $x_0 = 0$, $y_0 = 0$ and then easily calculate as many points as we wish in this sequence. Best of all would be to do the calculations by computer and use a graphing program to join up the points by line intervals and display the results on the screen or, using a printer, on paper.

The first example I ever saw of this construction is displayed in Figure 1. It is obtained from the function $f(n) = (\ln n)^4$ and was dubbed "the Loch Ness monster" by John Loxton (then in the School of Mathematics at UNSW, later at Macquarie University). We can analyse the shape of the "monster" along the lines given by John in his amusing article "Captain Cook and the Loch Ness Monster". Observe that the chain of line segments will be straight, or very slowly curving, if the change in angle between successive segments is close to zero or to any integer – since we are measuring the angle in full circles, increasing the angle by an integer does not change the direction of the interval. Similarly, the chain will turn back on itself and bunch up into a small region if the change in angle between successive intervals is approximately an integer plus a half. Now the difference in angle between the nth and $(n+1)$th intervals is

$$f(n+1) - f(n)\,,$$

and we can find a simpler approximate formula for this difference as follows. Remember that the derivative of a function is found by

$$f'(n) = \lim_{h\to 0} \frac{f(n+h) - f(n)}{h}\,.$$

Since the derivative is the limit of the quotient when h tends towards zero, we may expect that the derivative will approximately equal the quotient when h is small but not zero. Take $h = 1$. Then

$$f'(n) \simeq \frac{f(n+1) - f(n)}{1} = f(n+1) - f(n)\,.$$

In the case of the Loch Ness monster we find that the angle between the nth and $(n+1)$th intervals is approximately

$$f(n+1) - f(n) = f'(n) = \frac{d}{dn}\bigl(\ln n\bigr)^4 = \frac{4}{n}\bigl(\ln n\bigr)^3\,.$$

Drawing the graph of $y = f'(n)$ against n (do it! by computer or otherwise) shows that it has a maximum of about $y = 5.38$ somewhere near $n = 20$ and then decreases slowly towards zero. It passes through $y = \frac{1}{2}$ for the last time at $n = 4900$ approximately, and this accounts for the "blob" at the bottom of the picture. The preceding smooth section corresponds to $y \simeq 1$ at $n \simeq 1600$, the previous blob to $y \simeq 1\frac{1}{2}$, and so on.

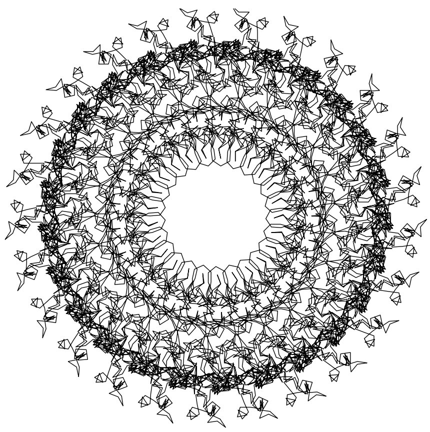

Figure 2: 7050 intervals, $f(n) = \dfrac{n}{25} + \dfrac{n^2}{12} + \dfrac{n^3}{94}$

Enough of the theory! Part of the fun of all this is just choosing a function more or less at random and seeing what you get. Try, for example,

$$f(n) = \frac{n}{d} + \frac{n^2}{m} + \frac{n^3}{y}\,.$$

Such a function usually (but not always) yields a pattern with d-fold rotational symmetry – that is, if you rotate the pattern by $360/d$ degrees around its centre you get exactly the same pattern. The denominators d, m, y may be any integers, for example the day, month and year of your birthday or some other special date. Figure 2 shows a Christmas card for 1994. OK, a bit out of date – but it's hard to complain when the traditional three kings have been increased to 25 kings bearing gifts as they travel around a circle. Figure 3 is the pattern I get for my own birthday. If you can work out the missing denominators, then don't forget to send me a card!

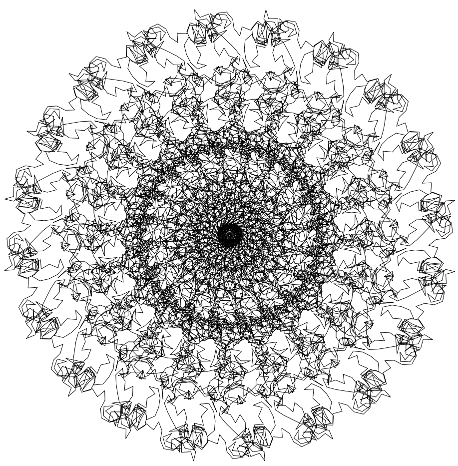

Figure 3: 10620 intervals, $f(n) = \dfrac{n}{?} + \dfrac{n^2}{?} + \dfrac{n^3}{?}$

A hint for constructing these patterns: the most interesting ones usually come from moderately large denominators with no common factors. You will often find that after a certain number of intervals, say L, you can stop as you have the whole pattern and further line segments merely repeat those you have already. The number L can be taken as the least common multiple of d, m, y, and sometimes even smaller.

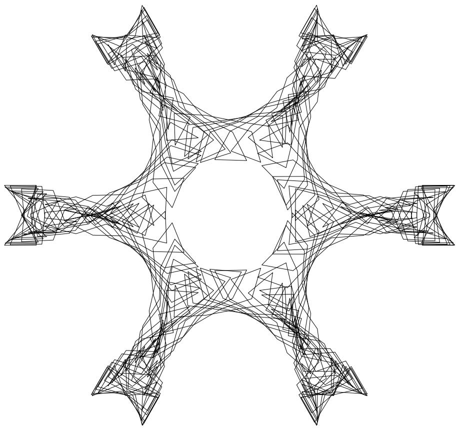

Figure 4: 1000 intervals, $f(n) = \dfrac{n}{6} + \cos n$

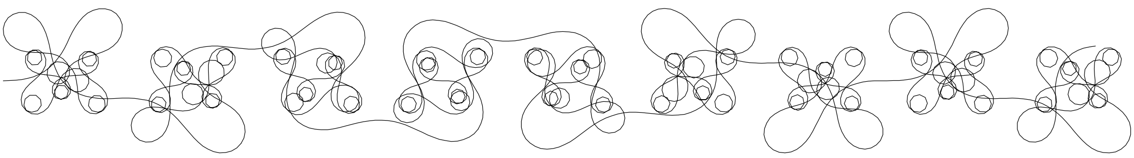

Figure 5: 2000 intervals, $f(n) = \sin(n/7)\sin(n/71)$

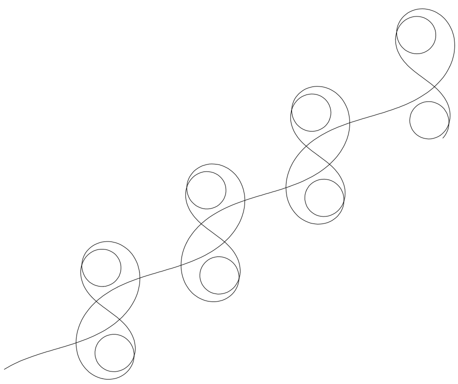

Figure 6: 1200 intervals, $f(n) = \dfrac{9 + 3\sin(n/50)}{8 - 3\sin(n/50)}$

In Figures 4–6, we see a few patterns which can be obtained from formulae involving trigonometric functions. These may be varied by altering the numbers, changing the sines and cosines to different functions, and so forth. If you have a computer available, see what you can come up with. Try some functions which haven't been mentioned in this article – not all functions give interesting patterns, but you won't know unless you try. As the advertising writers say, "the only limitation is your imagination"! Send a copy of any good patterns to *Parabola* and we'll publish them. Don't forget to include your name, school (if you are a student or teacher) and the number of intervals and formula you used.

And finally, if you were wondering how Lassie got into the title of this article? ... just stare at Figure 7 for a while.

Figure 7: 2005 intervals, $f(n) = \dfrac{n^3}{2005}$

Further reading

J.H. Loxton, *Captain Cook and the Loch Ness Monster*, James Cook Mathematical Notes **28** (1981), no. 3, 3060–3064. Included in https://www.maths.ed.ac.uk/cook/iss_no27.pdf.