The Radii of Hyper Circumsphere and Insphere through Equidistant Points

Sin Keong Tong

Three points $A$, $B$, and $C$ of equal distance from each other form an equilateral triangle in $\mathbb{R}^2$. The reader can verify that it is not possible to construct a figure with 4 equidistant points in $\mathbb{R}^2$, as the points form a rhombus where the long diagonal is of different length to the other 5 edges. To form a figure with 4 equidistant points $A, B, C, D \in \mathbb{R}^3$, we extend an equilateral triangle from each of the sides $AB$, $BC$, and $CA$, and join the points $D_1$, $D_2$, and $D_3$ as in Figure 1(a). The result is a regular tetrahedron.

Next, we extend from each of the faces of a regular tetrahedron $ABCD$ four new tetrahedra, as shown in Figure 2(a). By joining the points $E_1, E_2, E_3, E_4$ in $\mathbb{R}^4$, a figure with 5 equidistant points $A, B, C, D, E$ is obtained, as shown in Figure 2(b). The regular 5-simplex formed in this way has 10 edges, 10 faces (each of which is a regular triangle) and 5 tetrahedra.

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1Sin Keong Tong is a graduate student at the School of Mathematics and Statistics, UNSW Sydney, Australia.
(a) extending from a tetrahedron  

(b) 5-simplex

Figure 2: Constructing a 5-simplex from a tetrahedron

In general, $n+1$ equidistant points $P_1, \ldots, P_{n+1}$ in $\mathbb{R}^n$ form a regular $(n+1)$-simplex $S$. It is formed by extending from each $(n-2)$-dimensional face of a regular $n$-simplex in $\mathbb{R}^{n-1}$ a regular $n$-simplex and joining their $n$ new points into one new point.

The circumsphere is the hypersphere that goes through all of the $n+1$ points of $S$. The insphere is the largest hypersphere enclosed by these $n+1$ points.

Figure 3: Insphere and circumsphere of a tetrahedron

The purpose of this paper is to derive formulae, given by the Theorem 1 to follow, for the radius $R_n$ of the circumsphere and the radius $r_n$ of the insphere for regular $(n+1)$-simplices in $\mathbb{R}^n$. 

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**Example.** We will now demonstrate the technique in determining the radius of the circumsphere and insphere of the regular tetrahedron in $\mathbb{R}^3$, as shown in Figure 3, by extending from an equilateral triangle.

![Figure 4: The circumsphere and insphere of a triangle](image)

The points $A_2(-\frac{1}{2}, \frac{1}{\sqrt{12}})$, $B_2(\frac{1}{2}, \frac{1}{\sqrt{12}})$, $C_2(0, -\frac{1}{\sqrt{3}})$ in Figure 4 form an equilateral triangle with sides of length 1 satisfying the equation

$$x^2 + y^2 = \frac{1}{3} = \left( \frac{1}{\sqrt{3}} \right)^2.$$ 

Therefore, the circumsphere of the triangle has radius $R_2 = \frac{1}{\sqrt{3}}$. Also, then the radius of the insphere of the triangle is $r_2 = \frac{1}{\sqrt{12}}$ since it is the perpendicular distance from the origin $O$ to the line segment $A_2B_2$.

Now map the points $A_2, B_2, C_2$ to $A_3, B_3, C_3 \in \mathbb{R}^3$ by appending to each point a 0 as $z$-coordinate. Any point $P(0, 0, z)$ is equidistant to $A_3, B_3, C_3$, so, to form a regular tetrahedron, choose $z$ such that

$$z^2 + \left( \frac{1}{\sqrt{3}} \right)^2 = 1.$$ 

We see that $z = \frac{\sqrt{6}}{3}$, so the points

$$A_3\left( -\frac{1}{2}, \frac{1}{\sqrt{12}}, 0 \right), \quad B_3\left( -\frac{1}{2}, \frac{1}{\sqrt{12}}, 0 \right), \quad C_3\left( 0, -\frac{\sqrt{3}}{3}, 0 \right), \quad D_3\left( 0, 0, \frac{\sqrt{6}}{3} \right)$$

form a regular tetrahedron with edges of length 1 in $\mathbb{R}^3$.

The next step is to shift the points to a sphere with center at the origin. Let $P$ be the point $(0, 0, d)$ and note that $PA_3 = PB_3 = PC_3$. We will now find $P$ by considering the equation implied by $PC_3 = PD_3$:

$$\left( \frac{\sqrt{3}}{3} \right)^2 + d^2 = \left( d - \frac{\sqrt{6}}{3} \right)^2.$$
Solving this equation yields \( d = \frac{1}{\sqrt{24}} \), so \( P \) is the point \((0, 0, \frac{1}{\sqrt{24}})\). Since the point \( P \) lies on the \( z \)-axis, the distance \( d \) is the perpendicular distance from \( P \) to \( \triangle ABC \) (which lies in the \( x-y \) plane). By symmetry, it is also the perpendicular distance from \( P \) to \( \triangle BCD, \triangle CDA, \) and \( \triangle DAB \), and is therefore the insphere radius of the regular tetrahedron, namely

\[
r_3 = \frac{1}{\sqrt{24}}.
\]

By shifting each point down by \( \frac{1}{\sqrt{24}} \) in the \( z \) coordinate, the points

\[
A'_3\left(\frac{-1}{2}, \frac{\sqrt{3}}{6}, \frac{-1}{\sqrt{24}}\right), \quad B'_3\left(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{-1}{\sqrt{24}}\right), \quad C'_3\left(0, \frac{-\sqrt{3}}{3}, \frac{-1}{\sqrt{24}}\right), \quad D'_3\left(0, 0, \frac{\sqrt{6}}{4}\right)
\]

lie on the sphere

\[
x^2 + y^2 + z^2 = \frac{3}{8} = \left(\frac{\sqrt{3}}{8}\right)^2.
\]

These points form a regular tetrahedron centered around the origin. We see that the radius of circumsphere containing these points is

\[
R_3 = \frac{\sqrt{3}}{8}.
\]

**Theorem 1.** Let \( \{P_1, \ldots, P_{n+1}\} \) be \( n+1 \) points in \( \mathbb{R}^n \) with constant distance 1 between each two points. Then the radii \( R_n \) and \( r_n \) of the circumsphere and insphere of the \( n \)-simplex \( \{P_1, \ldots, P_{n+1}\} \) are, respectively,

\[
R_n = \sqrt{\frac{n}{2(n+1)}}, \quad r_n = \frac{1}{\sqrt{2n(n+1)}}.
\]

For instance,

\[
R_2 = \frac{1}{\sqrt{3}}, \quad r_2 = \frac{1}{\sqrt{12}}, \quad R_3 = \frac{\sqrt{3}}{8}, \quad r_2 = \frac{1}{\sqrt{24}},
\]

as we have seen in the example above.

**Proof.** The proof is by induction on \( n \). We have proven the cases \( n = 2, 3 \), so assume that the theorem is true for \( n \geq 3 \). Consider \( n+1 \) points \( \{P_1, \ldots, P_{n+1}\} \) in \( \mathbb{R}^n \) with constant distance 1 between each two points. By shifting and rotating these points, we can let the centre of the circumsphere and insphere of these points be the origin so that the point \( P_{n+1} = Z(n, -R_n) \), where

\[
Z(n, z) = (0, 0, \ldots, 0, z).
\]
Let the coordinates of each point be written as $P_k = (x_{k,1}, \ldots, x_{k,n})$. Then

$$\sum_{i=1}^{n} x_{k,i}^2 = R_n^2.$$  \hfill (3)

Map the points $P_1, \ldots, P_{n+1} \in \mathbb{R}^n$ to the points $Q_1, \ldots, Q_{n+1} \in \mathbb{R}^{n+1}$ by appending a 0 in coordinate position $n+1$.

Define

$$Q_{n+2} = Z(n+1, \sqrt{1 - R_n^2}).$$

Then, for each $k = 1, \ldots, n+1$,

$$|Q_{n+2}Q_k|^2 = R_n^2 + 1 - R_n^2 = 1.$$  \hfill (4)

Therefore, the points $Q_1, \ldots, Q_{n+1}$ are each at distance 1 from $Q_{n+2}$, so these $n+2$ points form a regular $(n+1)$-simplex in $\mathbb{R}^{n+1}$.

Let $C = Z(n+1, d)$ for some $d$. For each $k = 1, \ldots, n+1$, identity (3) implies that

$$|CQ_k|^2 = R_n^2 + d,$$

so $C$ is equidistant to the points $Q_1, \ldots, Q_{n+1}$.

We wish to determine $d$ so that $C$ also has this distance to $Q_{n+2}$. Since

$$R_n^2 + d^2 = \left(d - \sqrt{1 - R_n^2}\right)^2,$$

we see that

$$d = \frac{1 - 2R_n^2}{2\sqrt{1 - R_n^2}}.$$  \hfill (5)

Here, $d$ is the perpendicular distance from the centre of the circle at $C$ to the hyperplane containing $\{Q_1, \ldots, Q_{n+2}\}$. Because of the symmetry of a regular $(n+1)$-simplex, $d$ is also, for each $k = 1, \ldots, n+2$, the projection of each point to the hyperplane through $\{Q_1, \ldots, Q_{n+2}\} \setminus Q_k$. Therefore, $C$ is the insphere to the points $Q_1, \ldots, Q_{n+2}$, and so $r_{n+1} = d$. By (4),

$$r_{n+1} = \frac{1 - \frac{2(n-1)}{2n+2}}{2\sqrt{1 - \frac{n}{2n+2}}} = \frac{1}{\sqrt{2(n+1)(n+2)}}.$$  \hfill (6)

Since $Z(n+1, r_{n+1})$ is the centre of the circle through the points $Q_1, \ldots, Q_{n+2}$, the radius of the circumsphere is given by distance between $Z(n+1, r_{n+1})$ and $Z(n+1, \sqrt{1 - R_n^2})$. Therefore, by induction assumption and (3),

$$R_{n+1} = \sqrt{1 - R_n^2} - r_{n+1}$$

$$= \sqrt{1 - R_n^2} - \frac{1 - 2R_n^2}{2\sqrt{1 - R_n^2}}$$

$$= \sqrt{1 - R_n^2} \left(1 - \frac{1 - R_n^2}{2(1 - R_n^2)}\right)$$

$$= \sqrt{1 - R_n^2} \left(1 - \frac{1 - R_n^2}{(n+1)(n+2)}\right).$$  \hfill (7)

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$$= \sqrt{1 - R_n^2} \left(1 - \frac{1 - R_n^2}{(n+1)(n+2)}\right).$$  \hfill (7)
Finally, map the points $Q_1, \ldots, Q_{n+2}$ to $R_1, \ldots, R_{n+2}$ by preserving the first $n$ coordinates and subtracting $d$ (calculated in (6)) from the $(n+1)^{th}$ coordinate. The points $R_1, \ldots, R_{n+2}$ are pairwise equidistant from each other, and therefore form the regular $n$-simplex in $\mathbb{R}^{n+1}$.

By induction, the proof is now complete. \hfill \Box

Theorem 1 provides a number of insights into the properties of a regular tetrahedron with $n+1$ vertices in the propositions below.

**Proposition 2.** By the construction, the centres of the insphere and circumsphere coincide.

**Proposition 3.** $R_n = nr_n$

**Proposition 4.** $\lim_{n \to \infty} R_n = \frac{1}{\sqrt{2}}$

**Proposition 5.** As $n \to \infty$, each point pair $P_iP_j$ subtend a right angle at the center $C$ (see Figure 5).

**Proof.** By the theorem above, we see that $R_n \to \frac{1}{\sqrt{2}}$ as $n \to \infty$. Therefore, Figure 5 and the cosine identity

$$\cos \theta = \frac{R_n^2 + R_n^2 - a^2}{2R_n^2}$$

imply that $\theta \to 90^\circ$ as $n \to \infty$. \hfill \Box

![Figure 5: Angle subtended by an edge to the centre of circumsphere](image)

Michelle, thank you for the song that summer sings.