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New relationships between Pythagorean Triples: Composition and Decomposition of triples

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This article shows that a new vista of relationships between Pythagorean triples emerges when triples are composed with one another. It proposes six Composition Laws for composing triples and generating all the infinitude of triples; and six Decomposition Laws for breaking up triples into other triples. Composition and decomposition reveals that, in the infinitely large set of Pythagorean triples, every triple is connected in a reversible parent/child relationship with every other triple; and any triple can generate any other when composed with its appropriate pair. Also, any triple can be seen as the result of a finite number of compositions as well as of an infinitely long chain of compositions. A comparison with parent/child relationships suggested by the linear transformations of Berggren-Price demonstrates the distinctiveness of the proposed composition/decomposition approach.

Introduction and overview

Pythagorean triples are natural number triples of the form (a, b, c) for which the square of the largest number, say c^2, is the sum of the squares of the smaller two, here $a^2 + b^2$. For instance, $(3, 4, 5)$ and $(3569, 2520, 4369)$ are Pythagorean triples since

$$3^2 + 4^2 = 5^2 \quad \text{and} \quad 3569^2 + 2520^2 = 4369^2\,.$$

The label "Pythagorean" derives from the fact that the numbers (a, b, c) satisfy Pythagoras' Theorem when represented as the side lengths of a right-angled triangle, with c as the hypotenuse length. Pythagorean triples have a long history, and tables of these triples were used extensively for easy and very accurate calculations by the Babylonians nearly 4.000 years ago [5].

If a, b, and c are relatively prime, with c being largest, then the Pythagorean triple (a, b, c) is a *primitive Pythagorean triple* (PPT). For such triples, c and either a or b is odd whereas the other is a multiple of 4. Pythagorean triples are infinite in number and, since antiquity, algorithms have been developed to generate them.

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Algorithms: Algebraic identities

The most common algorithm for generating triples is to take two relatively prime and distinct odd numbers s and t, where $s > t \geq 1$, and simply generate the PPT

$$\left(st, \frac{s^2 - t^2}{2}, \frac{s^2 + t^2}{2}\right).$$

This algorithm, or its variant $(m^2 - n^2, 2mn, m^2 + n^2)$, where one of m and n is even and the other odd, comes from Euclid. There is also the algorithm, attributed to Plato, where one starts from an even number uv and generates the triple, PPT or non-PPT, $\big((\frac{u}{2})^2 - v^2, uv, (\frac{u}{2})^2 + v^2\big)$ if $\frac{u}{2} > v$, or $\big(u^2 - (\frac{v}{2})^2, uv, u^2 + (\frac{v}{2})^2\big)$ if $u > \frac{v}{2}$. In fact, there are a host of algorithms for generating triples, like

$$(2n+1)^2 + \big(n(2n+1) + n\big)^2 = \big(n(2n+1) + n + 1\big)^2$$

given by Stifel (1544) [8], or

$$\big(n(4n+4) + (4n+3)\big)^2 + (4n+4)^2 = \big(n(4n+4) + (4n+3) + 2\big)^2$$

given by Ozanam (1694) [6], which are based on one integer variable instead of two, and generate some particular class, rather than all classes, of triples. Another famous algorithm, which makes use of the Fibonacci series of numbers (denoted by F_1, F_2, etc.), gives the triple $(F_n F_{n+3}, 2F_{n+1}F_{n+2}, F_{2n+3})$, where $n \geq 1$.

There may be other algorithms, too, based on some property of Pythagorean triples. For instance, I can develop an algorithm from the property $(c-a)(c-b)$ is twice the square of an integer.

It may be noted that all these algorithms are concerned entirely with generation of triples, and they employ some algebraic identity (in one or two variables) satisfying the Pythagorean equation $a^2 + b^2 = c^2$. *None of them seeks to explore any possible inter-relation of Pythagorean triples.*

Algorithms: From one triple to other triples

Can we generate all the infinitude of triples from just one given triple?

In 1934, Berggren [2] proposed three linear transformations which, when applied to a PPT, can produce three other PPTs. Later, Barning [1] and Hall [3] put these transformations into matrix form. In 2008, Price [7] proposed an entirely new set of linear transformations. In this way, parent-child relationships, in which one parent triple produces three children, have been developed; see also Karama [4] for more general matrix constructions. But while these linear transformations help in generating unique (non-recurrent) PPTs only from a given PPT, they only produce triples which represent right triangles of area greater than that of the original triple. Thus, $(3,4,5)$ is the "original" parent in the whole family of Pythagorean triples, and all the rest can be generated if and only if we start from $(3,4,5)$. Starting from $(15,8,17)$, for instance, we can never generate the triple $(3,4,5)$ by these transformations. (See *Section 8.1*.) So, whether we can generate all Pythagorean triples should depend on which triple we start from.

But can we not generate all triples, starting from any triple? Also, it may be asked: *Are these the only kind of relationships possible between Pythagorean triples?*

Composition and Decomposition
Much as we admire the elegent efficiency of these algorithms, we cannot but notice that *composition* of Pythagorean triples has not been considered as a process, nor their consequences studied.

In this discussion, my object is to apply *composition* on triples by utilizing a particular property of quadratic forms, and look at the consequences. By doing so, I shall propose six *Composition Laws* by which any two Pythagorean triples, same or distinct, primitive or non-primitive, can be composed to generate six other triples. By continuing this process interminably, all the infinitude of triples can be generated, regardless of which triple one starts from. I shall also demonstrate that any Pythagorean triple can be *decomposed*, or broken up, into six different pairs of triples according to six *Decomposition Laws*, and consequently into infinitely many pairs of triples. I shall also set up the *matrix equation* for both composition and decomposition.

Thus, beyond producing a powerful engine for generating all the infinitude of Pythagorean triples by starting from any triple, the present study will reveal a new vista of relationships between triples. It will be seen that under these relationships:

(i) the idea of an "original" parent in the family of Pythagorean triples is lost;

(ii) any two triples are bound in a reversible parent/child relationship; and

(iii) any triple can be seen as the result of a finite number of compositions as well as of an infinitely long chain of composition of triples.

I have focussed on the composition and decomposition of PPTs, and into PPTs, only.

1 Composition of Triples

I propose to represent Pythagorean triples as two kinds of matrices, named *generating matrices* and *composing matrices*.

1.1 Generating Matrices

Consider any Pythagorean triple (a, b, c) (i.e., $a^2 + b^2 = c^2$), and define the following 3×3 ***generating matrices***:

$$G_{ax} = \begin{pmatrix} a & 0 & 0 \\ 0 & c & b \\ 0 & b & c \end{pmatrix} \qquad G_{by} = \begin{pmatrix} c & 0 & a \\ 0 & b & 0 \\ a & 0 & c \end{pmatrix} \qquad G_{cz} = \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$G_{bx} = \begin{pmatrix} b & 0 & 0 \\ 0 & c & a \\ 0 & a & c \end{pmatrix} \qquad G_{ay} = \begin{pmatrix} c & 0 & b \\ 0 & a & 0 \\ b & 0 & c \end{pmatrix} \qquad G_{zc} = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & c \end{pmatrix}$$

1.2 Composing Matrix and Triple Generation

Let $t = (x, y, z)$ also be a Pythagorean triple ($x^2 + y^2 = z^2$) and define the *Composing Matrix*

$$V = V(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

Then, it can be shown (in *Section 1.4*) that each of the generating matrices multiplied by the composing matrix generates a new Pythagorean triple in the form of another column vector. Thus, for instance,

$$G_{ax}V = \begin{pmatrix} a & 0 & 0 \\ 0 & c & b \\ 0 & b & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax \\ bz + cy \\ by + cz \end{pmatrix} ,$$

where $(ax)^2 + (bz + cy)^2 = (by + cz)^2$, a Pythagorean triple. More generally, the following matrix products all represent Pythagorean triples:

$$G_{ax}V = \begin{pmatrix} ax \\ bz + cy \\ by + cz \end{pmatrix} \qquad G_{by}V = \begin{pmatrix} az + cx \\ by \\ ax + cz \end{pmatrix} \qquad G_{cz}V = \begin{pmatrix} ax - by \\ ay + bx \\ cz \end{pmatrix}$$

$$G_{bx}V = \begin{pmatrix} bx \\ az + cy \\ ay + cz \end{pmatrix} \qquad G_{ay}V = \begin{pmatrix} bz + cx \\ ay \\ bx + cz \end{pmatrix} \qquad G_{zc}V = \begin{pmatrix} ax + by \\ ay - bx \\ cz \end{pmatrix}$$

1.3 General matrix equation

Dropping the subscripts, the general matrix equation is $GV = P$, where P represents a Pythagorean triple; or, $GV = dP_0$, where scalar $d = \gcd(\geq 1)$ of the rows of P, and P_0 a Primitive Pythagorean Triple (PPT). In all numerical examples used in this discussion the focus will lie primarily on PPTs.

1.4 Composition

Multiplication of the generating matrices with the composing matrix V represents composition of the triple (a, b, c) with the triple (x, y, z) in 6 different ways. Let the compositions corresponding to the matrix products $G_{ax}V$, $G_{bx}V$, etc. be named *Compositions* C_{ax}, C_{bx}, C_{by}, C_{ay}, C_{cz}, C_{zc}, respectively. The basis of the compositions is as follows.

C_{ax}: Multiplying the equations $a^2 = c^2 - b^2$ and $x^2 = z^2 - y^2$ gives

$$a^2x^2 = (c^2 - b^2)(z^2 - y^2) = (c + b)(z + y)(c - b)(z - y) = (by + cz)^2 - (bz + cy)^2 \,.$$

Therefore, $(ax)^2 + (bz + cy)^2 = (by + cz)^2$; that is, $(ax, bz + cy, by + cz)$ is also a Pythagorean triple.

Using $\bullet$ as the symbol for composition, this is composing $(a, b, c) \bullet (x, y, z)$ in a particular way, which I call *Composition* C_{ax}.

C_{bx}: Exchange a and b in C_{ax}, that is, compose $(b, a, c) \bullet (x, y, z)$ similarly as above.

C_{by}: Multiply $b^2 = c^2 - a^2$ and $y^2 = z^2 - x^2$, and proceed similarly as above.

C_{ay}: Exchange b and a in C_{by}; that is, compose $(b, a, c) \bullet (x, y, z)$ similarly as in C_{by}.

C_{cz}: Multiplying the equations $a^2 + b^2 = c^2$ and $x^2 + y^2 = z^2$, and by complex factorization on the left-hand side, I get,

$$(a + ib)(a - ib)(x + iy)(x - iy) = (cz)^2 \,;$$

or,

$$(a + ib)(x + iy)(a - ib)(x - iy) = (cz)^2 \,.$$

Therefore, $(ax - by)^2 + (ay + bx)^2 = (cz)^2$; that is, $(ax - by, ay + bx, cz)$ is also a Pythagorean triple. It does not matter if $(ax - by)$ is negative, what matters in a triple is the absolute value $|ax - by|$.

C_{zc}: It arises from the identity $(ax - by)^2 + (ay + bx)^2 = (ax + by)^2 + (ay - bx)^2$. So, $(ax + by, ay - bx, cz)$ is yet another Pythagorean triple.

1.5 The quadratic form

The main inspiration behind the compositions is that the quadratic form $(p^2 \pm Nq^2)$ is closed under multiplication. As any Pythagorean equation contains a quadratic form of this kind, it is this property that makes their solutions *composable*.

1.6 Infinitely many triples

The fact that triples can be generated by composition implies that the Pythagorean equation has infinitely many Diophantine solutions, that is, Pythagorean triples are infinite in number, and can be generated by an endless process of composition. For instance, let composing triple t with itself by any of the compositions generate triple t_1. Then compose t_1 with t to get a triple t_2; and t_2 with itself to yield the triple t_3. Applying all the 6 compositions together will lead to even greater proliferation. Thus continuing *ad infinitum*, infinitely many triples will be generated.

1.7 Recurrences

As the process of composition is continued, and all possible compositions are admitted, and all non-primitive triples are reduced to PPTs, there will be recurrences of the same triple, but that will happen only when certain conditions obtain and will never exhaust the possibility of generating new triples. There will, in fact, be *infinitely many new triples and infinitely many recurrences* (see *Section 3*).

1.8 Composition Laws

I therefore propose six ***Composition Laws*** as follows.

Any two Pythagorean triples (a, b, c) and (x, y, z), same or distinct, primitive or non-primitive, can be composed in 6 different ways to generate the following 6 triples:

$$\begin{array}{lll} C_{ax}: (ax, bz+cy, by+cz) & C_{by}: (az+cx, by, ax+cz) & C_{cz}: (ax-by, ay+bx, cz) \\ C_{bx}: (bx, az+cy, ay+cz) & C_{ay}: (bz+cx, ay, bx+cz) & C_{zc}: (ax+by, ay-bx, cz) \end{array}$$

By continuing the process of composition *ad infinitum* among the triples generated at every stage, all the infinitude of Pythagorean triples will be generated. By dividing each triple by the $\gcd$ (≥ 1) of its members, all the infinitude of Primitive Pythagorean Triples (PPT) will be generated. These formulations can be easily transformed into matrices as I have done in *Sections 1.1* and *1.2*.

1.9 Notation

Let composition be denoted generally as $(a, b, c) \bullet (x, y, z)$, and the specific composition modes as $C_{ax}[(a, b, c) \bullet (x, y, z)]$, $C_{bx}[(a, b, c) \bullet (x, y, z)]$, etc. The subscripts of G and C are ax, bx, etc. because, as will be seen later, these product terms characterize and play a vital role in each composition.

2 Proliferation of triples

The *Composition Laws* can be used as a powerful engine for generating Pythagorean triples. In the table below, I compose triples t_1 and t_2 to construct triples P_{ax}, P_{bx}, P_{by}, P_{ay}, P_{cz}, P_{zc} and then reduce them to their corresponding PPTs P_0 by dividing by their respective $\gcd$s. It is PPTs only that are used in all compositions.

t_1	t_2	P_{ax}	$\frac{P_{ax}}{\gcd}$	P_{bx}	$\frac{P_{bx}}{\gcd}$	P_{by}	$\frac{P_{by}}{\gcd}$	P_{ay}	$\frac{P_{ay}}{\gcd}$	P_{cz}	$\frac{P_{cz}}{\gcd}$	P_{zc}	$\frac{P_{zc}}{\gcd}$
15	45	675	3	360	360	1560	195	1189	1189	451	451	899	899
8	28	900	4	1271	1271	224	28	420	420	780	780	60	60
17	53	1125	5	1321	1321	1576	197	1261	1261	901	901	901	901
	gcd	225	1	1	1	8	1	1	1	1	1	1	1
15	3	45	5	24	24	126	63	91	91	13	13	77	77
8	4	108	12	143	143	32	16	60	60	84	84	36	36
17	5	117	13	145	145	130	65	109	109	85	85	85	85
	gcd	9	1	1	1	2	1	1	1	1	1	1	1
5	3	15	15	36	36	64	4	99	99	-33	-33	63	63
12	4	112	112	77	77	48	3	20	20	56	56	16	16
13	5	113	113	85	85	80	5	101	101	65	65	65	65
	gcd	1	1	1	1	16	1	1	1	1	1	1	1
13	3	39	39	252	28	320	20	675	675	-297	-297	375	15
84	4	760	760	405	45	336	21	52	52	304	304	200	8
85	5	761	761	477	53	464	29	677	677	425	425	425	17
	gcd	1	1	9	1	16	1	1	1	1	1	25	1
20	3	60	60	63	7	187	187	192	12	-24	-24	144	144
21	4	221	221	216	24	84	84	80	5	143	143	17	17
29	5	229	229	225	25	205	205	208	13	145	145	145	145
	gcd	1	1	9	1	1	1	16	1	1	1	1	1
15	360	5400	24	2880	1440	25935	25935	16688	16688	-4768	-4768	15568	15568
8	1271	32175	143	41422	20711	10168	10168	19065	19065	21945	21945	16185	16185
17	1321	32625	145	41522	20761	27857	27857	25337	25337	22457	22457	22457	22457
	gcd	225	1	2	1	1	1	1	1	1	1	1	1

Table 1: *Composing* $t_1 \bullet t_2 \to P_0$

The table shows that, for instance,

$$C_{ax}[(15,8,17) \bullet (45,28,53)] = (675,900,1125) \to (3,4,5)\,;$$

that is, $(15,8,17)$ and $(45,28,53)$ is composed (via C_{ax}) to give $(675,900,1125)$ which in turn reduces to $(3,4,5)$. I then use $(3,4,5)$ to generate the triples $(5,12,13)$, $(7,24,25)$, $(13,84,85)$, $(20,21,29)$ and many more from my compositions.

(i) *Identical Recurrences.* Note that compositions can recur:

$$C_{cz}[(15,8,17) \bullet (3,4,5)] = (13,84,85)$$
$$C_{zc}[(13,84,85) \bullet (3,4,5)] = (375,200,475) \to (15,8,17) \quad \text{(the initial triple)}$$
$$C_{by}[(15,8,17) \bullet (3,4,5)] = (126,32,130) \to (63,16,65)$$
$$C_{ax}[(15,8,17) \bullet (3,4,5)] = (45,108,117) \to (5,12,13)$$
$$C_{zc}[(5,12,13) \bullet (3,4,5)] \to (63,16,65) \quad \text{(as also derived above)}$$

Also,

$$C_{bx}[(15,8,17) \bullet C_{ax}[(15,8,17) \bullet (45,28,53)]] = C_{bx}[(15,8,17) \bullet (3,4,5)] = (24,143,145)$$
$$C_{ax}[(15,8,17) \bullet C_{bx}[(15,8,17) \bullet (45,28,53)]] = C_{ax}[(15,8,17) \bullet (360,1271,1321)]$$
$$\to (24,143,145) \quad \text{(as above)}\,.$$

(See note on *Commutative Pairs* below.)

(ii) *There are also recurrences with the order of first two terms reversed:*

$$C_{ax}[(15,8,17) \bullet (45,28,53)] \to (3,4,5)$$
$$C_{ax}[(15,8,17) \bullet (3,4,5)] \to (5,12,13)$$
$$C_{by}[(5,12,13) \bullet (3,4,5)] = (64,48,80) \to (4,3,5)\,.$$

Thus, we have $(3,4,5)$ and $(4,3,5)$. Similarly, there are $(5,12,13)$ and $(12,5,13)$, and $(45,28,53)$ and $(28,45,53)$.

3 Properties of Composition

It has been seen in Table 1 that there are recurrences among the triples. This leads to the question: *Will recurrences eventually exhaust the generation of new triples?* The discussion on the properties of composition, apart from the academic interest involved, is motivated mainly by this question. In view of the kind of relationships that is being sought, it is important that the inexhaustibility of the triple generation process is established. Most of the examples for reference are taken from Table 1.

1. **PPTs and non-PPTs.** Both PPTs and non-PPTs are generated by the compositions. As PPTs are more fundamental, the non-PPTs have all been reduced to their corresponding PPTs in this study. In terms of matrices, this amounts to scalar multiplication as indicated in the matrix equation $GV = dP_0$ (see *Section 1.3*).

2. $b = c - 1$ **and** $b < c - 1$. PPTs are of two kinds, one in which the even member b equals $c-1$ and the other in which $b < c-1$. Both kinds are generated by composition.

3. **Trivial triple.** For $t = (a,b,c)$, the composition $G_{zc}(t)V(t)$ is the triple $(c^2, 0, c^2)$ which reduces to $(1,0,1)$, a "trivial" triple. It will be subsequently ignored.

4. **Commutativity.**
(i) *Reversal of composition sequence.* For $t_1 = (a,b,c)$ and $t_2 = (x,y,z)$,

$$C_{ax}(t_1 \bullet t_2) = C_{ax}(t_2 \bullet t_1)\,;$$
$$C_{by}(t_1 \bullet t_2) = C_{by}(t_2 \bullet t_1)\,;$$
$$C_{cz}(t_1 \bullet t_2) = C_{cz}(t_2 \bullet t_1)\,;$$

that is, the compositions C_{ax}, C_{by}, C_{cz} are *commutative* under reversal of sequence. C_{zc} is also commutative if negative signs are ignored in the results.

However, $C_{bx}(t_1 \bullet t_2) = C_{ay}(t_2 \bullet t_1)$ (with first two terms reversed); that is, C_{bx} and C_{ay} are swapped under reversal of sequence. *Thus, under a particular sequence of compositions and its reverse we remain within the same set of results.*

(ii) *Commutative pairs.* Note that the following matrix products are commutative,

$$G_{ax}G_{bx} = G_{bx}G_{ax}\,;$$
$$G_{by}G_{ay} = G_{ay}G_{by}\,;$$
$$G_{cz}G_{zc} = G_{zc}G_{cz}\,.$$

This implies that

$$C_{ax}[t_1 \bullet \{C_{bx}(t_1 \bullet t_2)\}] = C_{bx}[t_1 \bullet \{C_{ax}(t_1 \bullet t_2)\}]\,;$$

that is, the compositions (C_{ax}, C_{bx}) form a kind of *commutative pair*. For instance, consider the recurrence of triple $(24, 143, 145)$ in Table 1; see *Identical Recurrences* in Section 2 above. Similarly, the compositions (C_{by}, C_{ay}) and (C_{cz}, C_{zc}) also form *commutative pairs*.

5. **Chain of compositions: Associativity.** By definition,

$$C_{ax}[(C_{ax}(t_1 \bullet t_2)) \bullet (C_{ax}(t_3 \bullet t_4))] = C_{ax}[C_{ax}(C_{ax}(t_1 \bullet t_2)) \bullet t_3)] \bullet t_4]\,;$$

that is, a chain of compositions under C_{ax} is *associative*. Similarly, compositions under C_{by}, C_{cz} are also *associative*. Compositions under C_{bx}, C_{ay}, C_{zc} are non-associative.

6. **Reversal of order: first two terms.** Table 1 contained recurrences of triples with the order of the first two terms reversed; such as, $(3, 4, 5)$, $(4, 3, 5)$, or $(5, 12, 13)$, $(12, 5, 13)$. *What happens if these "reversed" triples are used in subsequent compositions?*

By the definitions, it can be seen that, when the order of the first two terms are reversed, pairs of compositions exchange results as follows:

(i) Comparing $(a, b, c) \bullet (x, y, z)$ and $(b, a, c) \bullet (x, y, z)$:
Compositions (C_{ax}, C_{bx}) exchange results, and so do (C_{by}, C_{ay}). Meanwhile, (C_{cz}, C_{zc}) exchange results if negative signs are ignored and the first two terms are reversed.

(ii) Comparing $(a, b, c) \bullet (x, y, z)$ and $(a, b, c) \bullet (y, x, z)$:
Compositions (C_{ax}, C_{ay}) exchange results (with their first two terms reversed); so do (C_{bx}, C_{by}), and (C_{cz}, C_{zc}) also exchange results with first two terms reversed.

(iii) Comparing $(a, b, c) \bullet (x, y, z)$ and $(b, a, c) \bullet (y, x, z)$:
Compositions (C_{by}, C_{ax}) exchange results (with their first two terms reversed), as do (C_{bx}, C_{ay}) (with first two terms reversed). However, (C_{cz}, C_{zc}) remain unchanged (with negative signs ignored) under reversal.

Thus again, notwithstanding negative values and reversed order of first two terms, *we remain effectively within the same set of results as in compositions* $(a, b, c) \bullet (x, y, z)$.

7. **Negative values.** Keeping or ignoring negative signs produces quite different consequences. For instance, $C_{ax}[(4081, -1560, 4369) \bullet (15, 8, 17)]$ gives $(61215, 8432, 61793)$, but $C_{ax}[(4081, 1560, 4369) \bullet (15, 8, 17)]$ gives $(61215, 61472, 86753)$. So, strictly speaking, retaining the negative signs ought to be a more authentic process.

8. **Non-PPTs: the conditions.** Table 1 shows that non-PPTs are generated, which can be reduced to PPTs. For example, $C_{by}[(15, 8, 17) \bullet (3, 4, 5)] = (126, 32, 130) \to (63, 16, 65)$. The same PPT can be generated directly from some other composition, and that contributes to recurrences of triples; for example, $C_{zc}[(5, 12, 13) \bullet (3, 4, 5)] = (63, 16, 65)$.

It can be seen that in each of the triples generated by composition there is a product term $\{ax, bx, by, ay, cz\}$, and the other two terms are sums/differences of products, like $(bz + cy), (ax - by)$. It will be shown here that non-PPTs are generated only when there is a common divisor $d > 1$ between the two factors of the product term; for example, the term ax may have $d > 1$, where $d | \gcd(a, x)$. *This is the essential, but not the only, condition*; certain other conditions are to be satisfied in order to generate a non-PPT.

In the discussion below I shall look into these other conditions. Consider the PPTs (a, b, c) and (x, y, z) where b and y are even.

Composition C_{ax}: Here, $C_{ax}[(a, b, c) \bullet (x, y, z)] = (ax, bz + cy, by + cz)$. Let $d > 1$ be a common divisor of a and x, not necessarily $\gcd(a, x)$. Then ax is divisible by d^2. Set $a = dma_0$ and $x = dnx_0$, where $\gcd(dm, a_0) = \gcd(dn, x_0) = 1$. Then either

(i)

$$b = \frac{1}{2}(d^2m^2 - {a_0}^2) \qquad y = \frac{1}{2}({x_0}^2 - n^2d^2)$$
$$c = \frac{1}{2}(d^2m^2 + {a_0}^2) \qquad z = \frac{1}{2}({x_0}^2 + n^2d^2)$$

or

(ii)

$$b = \frac{1}{2}(d^2m^2 - {a_0}^2) \qquad y = \frac{1}{2}(n^2d^2 - {x_0}^2)$$
$$c = \frac{1}{2}(d^2m^2 + {a_0}^2) \qquad z = \frac{1}{2}(n^2d^2 + {x_0}^2)\,.$$

For (i), $ax = d^2mna_0x_0$ and

$$bz + cy = \frac{1}{2}d^2(m^2{x_0}^2 - n^2{a_0}^2)$$
$$by + cz = \frac{1}{2}d^2(m^2{x_0}^2 + n^2{a_0}^2)\,.$$

As a and x are both odd, d, m, n, a_0, x_0 are odd, so $m^2x_0^2 - n^2a_0^2 = 2p$ is even.

So in the composed triple, $\gcd(d^2mna_0x_0, d^2p, d^2(p + n^2a_0^2) \geq d^2$.

For (ii), $ax = d^2mna_0x_0$ and

$$bz + cy = \frac{1}{2}(d^4m^2n^2 - a_0^2x_0^2)$$
$$by + cz = \frac{1}{2}(d^4m^2n^2 + a_0^2x_0^2)\,.$$

Let $q = \gcd\big(d^2mna_0x_0, \frac{1}{2}(d^4m^2n^2 - a_0^2x_0^2), \frac{1}{2}(d^4m^2n^2 + a_0^2x_0^2)\big)$. Then

$$2q\Big|\big((d^4m^2n^2 - a_0^2x_0^2) \pm (d^4m^2n^2 + a_0^2x_0^2)\big)\,,$$

so $q|(dm)^2(dn)^2$ and $q|(a_0^2x_0^2)$.

But $\gcd(dm, a_0) = \gcd(dn, x_0) = 1$, so $\gcd\big((dm)^2(dn)^2, a_0^2x_0^2\big) = 1$; that is, $q = 1$. Therefore, for (ii), the triple $(ax, bz + cy, by + cz)$ is a PPT.

Therefore, the necessary condition for a non-PPT to arise from C_{ax} is that $d > 1$, $dm > a_0$, and $x_0 > dn$;
if these conditions are not met, then the triple in question is a PPT.

For instance, $C_{ax}[(15, 8, 17) \bullet (45, 28, 53)] = (675, 900, 1125)$ is a non-PPT with

$$\gcd(675, 900, 1125) = 15^2 = 3^25^2$$

and $d = 5$ (see Table 1). Similarly, $C_{ax}[(3, 4, 5) \bullet (21, 220, 221)] = (63, 1984, 1985)$ is a PPT; while $C_{ax}[(3, 4, 5) \bullet (21, 20, 29)] = (63, 216, 225)$ is a non-PPT with $\gcd(63, 216, 225) = 3^2$.

Composition C_{bx}: Now consider $C_{bx}[(a, b, c) \bullet (x, y, z)] = (bx, az + cy, ay + cz)$.
Let $b = dmb_0$ and $x = dnx_0$ with b even and x odd; then d, n, and x_0 must be odd, and $\gcd(dn, x_0) = 1$. Then either

(i)

$$a = b_0^2 - \left(\frac{md}{2}\right)^2 \qquad y = \frac{1}{2}(d^2n^2 - {x_0}^2)$$
$$c = b_0^2 + \left(\frac{md}{2}\right)^2 \qquad z = \frac{1}{2}(d^2n^2 + {x_0}^2)$$

or

(ii)

$$a = b_0^2 - \left(\frac{md}{2}\right)^2 \qquad y = \frac{1}{2}(x_0^2 - d^2n^2)$$
$$c = b_0^2 + \left(\frac{md}{2}\right)^2 \qquad z = \frac{1}{2}(x_0^2 + d^2n^2)\,.$$

So, in both cases md is even and, since d is odd, m must be even.

For (i), if $b_0 > \frac{md}{2}$, $dn > x_0$ and m is even, then

$$(bx, az + cy, ay + cz) = \left(d^2mnb_0x_0, \frac{1}{4}d^2(4b_0^2n^2 - m^2x_0^2), \frac{1}{4}d^2(4b_0^2n^2 + m^2x_0^2)\right).$$

So, the greatest common divisor of the resultant triple is equal to at least d^2. Thus, $C_{bx}[(11, 60, 61) \bullet (15, 8, 17)] = (900, 675, 1125)$, a non-PPT with

$$\gcd(900, 675, 1125) = 15^2\,.$$

Here $d = 5$, but the greatest common divisor of the resultant triple is 3^25^2.

For (ii), $bx = d^2mnb_0x_0$ and

$$az + cy = \left(\frac{d^2mn}{2}\right)^2 - (b_0x_0)^2$$
$$ay + cz = \left(\frac{d^2mn}{2}\right)^2 + (b_0x_0)^2\,.$$

As d, n, and x_0 are odd, b_0 is odd as well, and $4|m$ because $az + cy$ and $ay + cz$ are both odd. Also, as b_0, x_0 are odd, b_0x_0 is odd too. Let

$$q = \gcd(d^2mnb_0x_0, \left(\frac{d^2mn}{2}\right)^2 - (b_0x_0)^2, \left(\frac{d^2mn}{2}\right)^2 + (b_0x_0)^2)\,.$$

Then

$$q\big|\big((\frac{d^2mn}{2})^2 - (b_0x_0)^2\big) \pm \big((\frac{d^2mn}{2})^2 + (b_0x_0)^2\big)\,;$$

therefore, $q|(d^2mn)^2$, and so $q|2(d^2mn)^2$; also, $q|2(b_0x_0)^2$. However, $\gcd(dm, b_0) = 1$ and $\gcd(dn, a_0) = 1$, so $\gcd(d^2mn, b_0x_0) = 1$. Therefore, $q|2(d^2mn)^2$ and $q|2(b_0x_0)^2$ imply that $q = 1$ or 2. This is a contradiction because the greatest common divisor q divides the odd terms $az + cy$ and $ay + cz$, and therefore cannot be 2. So, $q = 1$. Thus,

$$C_{bx}[(5, 12, 13) \bullet (3, 4, 5)] = (36, 77, 85)\,,$$

a PPT.

Thus, the necessary condition for a non-PPT to arise from C_{bx} is that $d > 1$, $b_0 > \frac{md}{2}$, $dn > x_0$, and that m is even;
if these conditions are not satisfied, then the greatest common divisor is 1.

Composition C_{ay}:
The case for $C_{ay}[(a, b, c) \bullet (x, y, z)] = (bz + cx, ay, bx + cz)$ is similar to that of C_{bx}.

Composition C_{by}: In $C_{by}[(a, b, c) \bullet (x, y, z)] = (az + cx, by, ax + cz)$ the terms are all even; so, 2 is always a common divisor. Let $b = dmb_0$ and $y = dny_0$. Here, if

$$a = b_0^2 - \left(\frac{md}{2}\right)^2 \qquad x = (nd)^2 - \left(\frac{y_0}{2}\right)^2$$
$$c = b_0^2 + \left(\frac{md}{2}\right)^2 \qquad z = (nd)^2 + \left(\frac{y_0}{2}\right)^2$$

then $by = d^2mnb_0y_0$ and

$$az + cx = \frac{1}{8}d^2(16n^2b_0^2 - m^2y_0^2$$
$$ax + cz = \frac{1}{8}d^2(16n^2b_0^2 + m^2y_0^2)\,.$$

So, $\gcd(az + cx, by, ax + cz) = 2p$, where $p > 1$.

Here, the greatest common divisor of the resultant triple is usually 2; and, if the greatest common divisor is greater than 2, then the necessary conditions are that
md and y_0 are even, and $b_0 > \frac{md}{2}$ and $nd > \frac{y_0}{2}$.

So, $\gcd(az + cx, by, ax + cz) \geq 2$. Thus,

$$C_{by}[(7, 24, 25) \bullet (35, 12, 37)] = (1134, 288, 1170) \quad (\gcd = 18)$$
$$C_{by}[(5, 12, 13) \bullet (7, 24, 25)] = (216, 288, 360) \quad (\gcd = 72)$$

but

$$C_{by}[(5, 12, 13) \bullet (9, 40, 41)] = (322, 480, 578) \quad (\gcd = 2)\,.$$

Compositions C_{cz} and C_{zc}: If $\gcd(c, z) = d$ for PPTs (a, b, c) and (x, y, z), then C_{cz} gives $(ax - by, ay + bx, d^2c_0z_0)$ and C_{zc} gives $(ax + by, ay - bx, d^2c_0z_0)$ where $\gcd(d, c_0, z_0) = 1$. As $\gcd(a, b, c) = 1$ and $\gcd(x, y, z) = 1$, in either case the greatest common divisor of the resultant triple can be d^2. If the greatest common divisor is $d^2 > 1$ for C_{cz}, then the greatest common divisor for C_{zc} is 1; and vice versa.

So, a non-PPT with greatest common divisor d^2 will be generated by either C_{cz} or C_{zc} but never both.

Thus, for $(5, 12, 13) \bullet (33, 56, 65)$, C_{cz} generates $(-507, 676, 845)$ with greatest common divisor 13^2; and C_{zc} generates $(837, 116, 845)$, a PPT. But for $(5, 12, 13) \bullet (63, 16, 65)$, C_{cz} generates $(123, 836, 845)$, a PPT; while C_{zc} gives $(507, 676, 845)$ which has greatest common divisor 13^2.

However, there can be the very special case of PPTs (a, b, c) and (x, y, c), the third terms being equal, as in $(33, 56, 65)$ and $(63, 16, 65)$. The third term in both C_{cz} and C_{zc} is c^2. Obviously, the greatest common divisor of any of the resultant triples cannot be c^2. Let $c = mn$ where $\gcd(m, n) = 1$. Therefore, if $\gcd(ax - by, ay + bx, m^2n^2) = m^2$, then $\gcd(ax + by, ay - bx, m^2n^2) = n^2$.

Thus, $C_{cz}[63, 16, 65) \bullet (33, 56, 65)] = (1183, 4056, 4225)$ with

$$\gcd(1183, 4056, 4225) = 13^2\,;$$

whereas $C_{zc}[63, 16, 65) \bullet (33, 56, 65)] = (2975, 4000, 4225)$ with

$$\gcd(2975, 4000, 4225) = 5^2\,.$$

The other special case $C_{zc}[(a, b, c) \bullet (a, b, c)] = (c^2, 0, c^2) \to (1, 0, 1)$ has been discussed above in *Trivial triples* (Property 3).

9. **General expression for the greatest common divisor.** In the item above, I have studied only the conditions when the resultant triple has greatest common divisor greater than 1. In general, as will be shown below in *Section 6* on decomposition, if (a, b, c) and (x, y, z) are both PPTs, then for any composition $(a, b, c) \bullet (x, y, z) \to (p, q, r)$ that is a PPT, we have that the greatest common divisor equals

$$\frac{a^2}{k_{ax}} \text{ (for } C_{ax}) \qquad \frac{b^2}{k_{by}} \text{ (for } C_{by}) \qquad \frac{c^2}{k_{cz}} \text{ (for } C_{cz})$$

$$\tfrac{b^2}{k_{bx}} \text{ (for } C_{bx}) \qquad \frac{a^2}{k_{ay}} \text{ (for } C_{ay}) \qquad \frac{c^2}{k_{zc}} \text{ (for } C_{zc})$$

where each of k_{ax}, k_{bx}, k_{ay}, k_{by}, k_{cz}, and k_{zc} is any positive divisor of the numerator.

10. **Recurrence of triples.** The foregoing discussion shows that in a process of composition recurrence of triples will be caused by:

(i) Reversal of the order of the first two terms (Property 6);

(ii) Commutativity, wherever it exists (Property 4);

(iii) Associativity, wherever it exists (Property 5);

(iv) Most importantly, when non-PPTs are reduced to PPTs: that is, when the same PPT is generated at different stages in the composition process by composing different triples (Property 8).

Note that (ii) and (iii) can happen only when all possible sequences of compositions are tried during the process.

4 Infinitely many triples and infinitely many recurrences

Let us imagine a continuous composition process in which all possible sequences of composition are admitted, and all non-PPTs are reduced to PPTs; in other words, a process in which all factors that are likely to cause recurrences are operative. As the number of compositions increases, the number of recurrences is also likely to increase. The big question is: *Will there be a point of 'saturation' when triples are merely repeated and no new ones are generated?*

But we have seen that recurrences occur only when very specific conditions are met. So, at every composition there will be as much possibility of a new triple generation as of the recurrence of an old one. Therefore, the point of 'saturation' will never come to be, because while infinitely many new triples will be generated, there will also be infinitely many recurrences, and this will be a never-ending process.

What will happen is that:

(1) Over a finite range of compositions the number of recurrences of triples will vary according to the compositions chosen (that is, the effects of commutativity, etc. can be minimized by suitable choice of compositions)

(2) Over an infinite range of compositions, while infinitely many new triples will be generated, there will also be infinitely many recurrences; but that being a never-ending process, no point of 'saturation' when new triples cease to be generated will ever be reached.

It may be noted that the claim "infinitely many triples and infinitely many recurrences", which is rather intuitively explained at this juncture, will be firmly established at the end of the section on Decomposition.

5 What compositions reveal

It has been seen that any two Pythagorean triples can generate 6 other triples by composition. This implies the existence of a parent/child relationship between triples: every pair of triples produces 6 children.

On the other hand, in a never-ending process of compositions, every PPT is generated infinitely many times. Therefore, any PPT can be produced from infinitely many pairs of parents. This implies that the parent/child relationship between any two triples is *reversible*; that is, any PPT can produce any other if it pairs with the appropriate PPT by composition.

This also raises the question: *Which triples does a given triple originate from?* Or conversely, *which triples can a given triple be decomposed into?*

The implications of composition will be better revealed and this question will be dealt with in the discussion on decomposition, below in the following *Section 6*.

6 Decomposition of triples

My object is to find which triples can be composed to generate a given primitive triple

$$P_0 = \begin{pmatrix} p \\ q \\ r \end{pmatrix} ,$$

that is, which triples can P_0 be factorised, or decomposed, into. To do this, we might first consider a PPT (a, b, c) and solve the matrix equation $G_{ax}V = P_0$ where

$$G_{ax} = \begin{pmatrix} a & 0 & 0 \\ 0 & c & b \\ 0 & b & c \end{pmatrix} \qquad \text{and} \qquad V = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

to get

$$V = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{a^2} \begin{pmatrix} ap \\ cq - br \\ cr - bq \end{pmatrix} .$$

That is,

$$\begin{pmatrix} a & 0 & 0 \\ 0 & c & b \\ 0 & b & c \end{pmatrix} \begin{pmatrix} ap \\ cq - br \\ cr - bq \end{pmatrix} = a^2 \begin{pmatrix} p \\ q \\ r \end{pmatrix} .$$

Now, $(ap)^2 + (cq - br)^2 = (cr - bq)^2$, by substituting $p^2 + q^2 = r^2$ and $a^2 + b^2 = c^2$ suitably.

Therefore, whatever be the chosen PPT (a, b, c), the column vector

$$\begin{pmatrix} ap \\ cq - br \\ cr - bq \end{pmatrix}$$

also represents a Pythagorean triple. This implies that the matrix P representing a Pythagorean triple can be factorised into a 3×3 and 3×1 matrix (multiplied by a scalar) in infinite ways, where both the factors represent Pythagorean triples; that is, *any Pythagorean triple can be decomposed into infinitely many pairs of Pythagorean triples*.

The solution, however, needs a bit of generalising, as follows:

$$V = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{a^2} \begin{pmatrix} ap \\ cq - br \\ cr - bq \end{pmatrix} = \frac{k}{a^2} V_0 ,$$

where $k = \gcd(ap, cq - br, cr - bq) \geq 1$, and V_0 represents the PPT corresponding to V. Using the proper subscripts, the equation can be re-written as $G_{ax}V_{0(ax)} = \frac{a^2}{k_{ax}} P_0$.

6.1 Matrix equations

Similar to the above, I can formulate for all the compositions as the following *Matrix Equations*.

C_{ax}: $\quad G_{ax}V_{0(ax)} = \frac{a^2}{k_{ax}}P_0$ where $V_{0(ax)} = \frac{1}{k_{ax}}\begin{pmatrix} ap \\ cq-br \\ cr-bq \end{pmatrix}$ and $k_{ax} = \gcd(ap, cq-br, cr-bq)$;

C_{bx}: $\quad G_{bx}V_{0(bx)} = \frac{b^2}{k_{bx}}P_0$ where $V_{0(bx)} = \frac{1}{k_{bx}}\begin{pmatrix} bp \\ cq-ar \\ cr-aq \end{pmatrix}$ and $k_{bx} = \gcd(bp, cq-ar, cr-aq)$;

C_{by}: $\quad G_{by}V_{0(by)} = \frac{b^2}{k_{by}}P_0$ where $V_{0(by)} = \frac{1}{k_{by}}\begin{pmatrix} cp-ar \\ bq \\ cr-ap \end{pmatrix}$ and $k_{by} = \gcd(cp-ar, bq, cr-ap)$;

C_{ay}: $\quad G_{ay}V_{0(ay)} = \frac{a^2}{k_{ay}}P_0$ where $V_{0(ay)} = \frac{1}{k_{ay}}\begin{pmatrix} cp-br \\ aq \\ cr-bp \end{pmatrix}$ and $k_{ay} = \gcd(cp-br, aq, cr-bp)$;

C_{cz}: $\quad G_{cz}V_{0(cz)} = \frac{c^2}{k_{cz}}P_0$ where $V_{0(cz)} = \frac{1}{k_{cz}}\begin{pmatrix} ap+bq \\ aq-bp \\ cr \end{pmatrix}$ and $k_{cz} = \gcd(ap+bq, aq-bp, cr)$;

C_{zc}: $\quad G_{zc}V_{0(zc)} = \frac{c^2}{k_{zc}}P_0$ where $V_{0(zc)} = \frac{1}{k_{zc}}\begin{pmatrix} ap-bq \\ aq+bp \\ cr \end{pmatrix}$ and $k_{zc} = \gcd(ap-bq, aq+bp, cr)$.

6.2 Decomposition Laws

I can now set out the preceding formulations in the form of *Decomposition Laws*.

If (p, q, r) is a PPT, then it can be *decomposed* into a given PPT (a, b, c) and six other triples $\{t_{ax}, t_{bx}, t_{by}, t_{ay}, t_{cz}, t_{zc}\}$ in six different ways. That is, the PPT (p, q, r) can be generated by composing the PPT (a, b, c) with the triples $t_{ax}, t_{bx}, t_{by}, t_{ay}, t_{cz}, t_{zc}$ according to our six compositions C_{ax}, C_{bx}, C_{by}, C_{ay}, C_{cz}, C_{zc} respectively.

Given a PPT (a, b, c), any PPT (p, q, r) shall equal (can be decomposed as):

$\frac{1}{a^2}.C_{ax}[(a, b, c) \bullet t_{ax}]$, where $t_{ax} = [ap, (cq-br), (cr-bq)]$;

$\frac{1}{b^2}.C_{bx}[(a, b, c) \bullet t_{bx}]$, where $t_{bx} = [bp, (cq-ar), (cr-aq)]$;

$\frac{1}{b^2}.C_{by}[(a, b, c) \bullet t_{by}]$, where $t_{by} = [(cp-ar), bq, (cr-ap)]$;

$\frac{1}{a^2}.C_{ay}[(a, b, c) \bullet t_{ay}]$, where $t_{ay} = [(cp-br), aq, (cr-bp)]$;

$\frac{1}{c^2}.C_{cz}[(a, b, c) \bullet t_{cz}]$, where $t_{cz} = [(ap+bq), (aq-bp), cr]$;

$\frac{1}{c^2}.C_{zc}[(a, b, c) \bullet t_{zc}]$, where $t_{zc} = [(ap-bq), (aq+bp), cr]$.

Now we can replace (a, b, c) with t_{ax} and find which triples composed with t_{ax} will generate (p, q, r). Continuing this process *ad infinitum* we find that infinitely many pairs of triples can generate (p, q, r); that is, (p, q, r) can be decomposed into pairs of triples in infinitely many ways.

6.3 "Amicable" triples

As $(a, b, c) = T$ when composed with each of $t_{ax}, \ldots, t_{zc}$ individually generates the same triple (p, q, r), each of the pairs $(T, t_{ax}), (T, t_{bx}), (T, t_{by}), (T, t_{ay}), (T, t_{cz}), (T, t_{zc})$ can be named *Amicable Triples for generating* (p, q, r). Every triple has six *amicable pairs* for generating any particular triple.

There are infinitely many *amicable pairs* of triples for generating any given triple.

6.4 Decomposing the PPT $(15, 8, 17)$

In Table 1, I started with $(15, 8, 17)$ as one of the generating triples and got $(3, 4, 5)$ as my first result. In Table 2 below, by applying the decomposition formulas I have a glimpse of which pairs of PPTs can $(15, 8, 17)$ be decomposed into. In the first row I take $(3, 4, 5)$ as the generating triple and find its *amicable triples* for producing $(15, 8, 17)$.

G	V_{ax}	$\frac{V_{ax}}{\text{gcd}}$	V_{bx}	$\frac{V_{bx}}{\text{gcd}}$	V_{by}	$\frac{V_{by}}{\text{gcd}}$	V_{ay}	$\frac{V_{ay}}{\text{gcd}}$	V_{cz}	$\frac{V_{cz}}{\text{gcd}}$	V_{zc}	$\frac{V_{zc}}{\text{gcd}}$
3 4 5	45 -28 53	45 -28 53	60 -11 61	60 -11 61	24 32 40	3 4 5	7 24 25	7 24 25	77 -36 85	77 -36 85	13 84 85	13 84 85
gcd	1	1	1	1	8	1	1	1	1	1	1	1
45 -28 53	675 900 1125	3 4 5	-420 -341 541	-420 -341 541	30 -224 226	15 -112 113	1271 360 1321	1271 360 1321	451 780 901	451 780 901	899 -60 901	899 -60 901
gcd	225	1	1	1	2	1	1	1	1	1	1	1

Table 2: *Decomposing* $(15, 8, 17)$ *into* $(3, 4, 5)$ *and another triple, and into* $(45, -28, 53)$ *and another; finding Amicable Triples of triple* $(3, 4, 5)$ *and again of* $(45, -28, 53)$ *for generating* $(15, 8, 17)$.

It is found that, if $t_1 = (3, 4, 5)$, then

$$\begin{aligned}(15, 8, 17) &\leftarrow C_{ax}[t_1 \bullet (45, -28, 53)] \\ &\leftarrow C_{bx}[t_1 \bullet (60, -11, 61)] \\ &\leftarrow C_{by}[t_1 \bullet (3, 4, 5)] \\ &\leftarrow C_{ay}[t_1 \bullet (7, 24, 25)] \\ &\leftarrow C_{cz}[t_1 \bullet (77, -36, 85)] \\ &\leftarrow C_{zc}[t_1 \bullet (13, 84, 85)] \,.\end{aligned}$$

And similarly for $t_1 = (45, -28, 53)$.

6.5 Infinitely long chain of compositions

Consider triple $T = C_{ax}(t_1 \bullet t_2)$ and let $t_1 = C_{bx}(t_3 \bullet t_4)$ and $t_2 = C_{by}(t_5 \bullet t_6)$. Then

$$
\begin{aligned}
T &= C_{ax}[C_{bx}(t_3 \bullet t_4) \bullet C_{by}(t_5 \bullet t_6)] \\
&= C_{ax}[C_{bx}(t_3 \bullet t_4) \bullet C_{by}[t_5 \bullet C_{ay}(t_7 \bullet t_8)]]
\end{aligned}
$$

where $t_6 = C_{ay}(t_7 \bullet t_8)$, and so on. Thus, the chain of compositions generating T can go on increasing. As each of the component triples can be decomposed in infinitely many ways, the chain can be infinitely long.

6.6 Properties of Decomposition

Properties of decomposition are similar to those of composition. For instance, given the commutative nature of the compositions C_{ax}, C_{by}, C_{cz}, we will have

$$C_{ax}[(a,b,c) \bullet (ap, cq - br, cr - bq)] = C_{ax}[(ap, cq - br, cr - bq) \bullet (a,b,c)]\,,$$

and similarly for C_{by}, C_{cz}.

7 Infinitely many triples and infinitely many recurrences: Reviewed

Having studied decomposition, I can now look at the assertion "infinitely many triples and infinitely many recurrences" again.

It has been seen that every primitive triple can be decomposed into infinitely many distinct pairs of primitive triples. Each of these offspring or component PPTs can also be decomposed into infinitely many distinct pairs of PPTs. Therefore, in a process of infinitely many compositions every PPT will have infinitely many recurrences as the results of infinitely many compositions. As our proposed process starts with only two PPTs, same or distinct, and goes on composing the newly generated triples among themselves, the infinitely many distinct pairs of PPTs that can generate one particular PPT should all come from the triple generation process. So, unless infinitely many distinct triples are generated, there cannot be infinitely many recurrences of the same triple. But decomposition has shown that there will indeed be infinitely many recurrences of every triple. Therefore, in an endless process there will be infinitely many new and distinct triples and infinitely many recurrences.

8 Relationships between triples

8.1 The linear transformations of Berggren and Price

If (a, b, c) is a PPT, where a is odd and b even, the following are the transformations proposed by Berggren [2] and Price [7] independently.

Berggren: The PPTs (a_1, b_1, c_1), (a_2, b_2, c_2), and (a_3, b_3, c_3) are produced from (a, b, c) by the following transformations proposed by Berggren:

$$\begin{aligned} a_1 &= -a + 2b + 2c \qquad & b_1 &= -2a + b + 2c \qquad & c_1 &= -2a + 2b + 3c \\ a_2 &= +a + 2b + 2c & b_2 &= +2a + b + 2c & c_2 &= +2a + 2b + 3c \\ a_3 &= +a - 2b + 2c & b_3 &= +2a - b + 2c & c_3 &= +2a - 2b + 3c\,. \end{aligned}$$

Price: The PPTs (a_1, b_1, c_1), (a_2, b_2, c_2), and (a_3, b_3, c_3) are produced from (a, b, c) by the following transformations proposed by Price:

$$\begin{aligned} a_1 &= +2a + b - c \qquad & b_1 &= -2a + 2b + 2c \qquad & c_1 &= -2a + b + 3c \\ a_2 &= +2a + b + c & b_2 &= +2a - 2b + 2c & c_2 &= +2a - b + 3c \\ a_3 &= +2a - b + c & b_3 &= +2a + 2b + 2c & c_3 &= +2a + b + 3c\,. \end{aligned}$$

Parent	Child $Bn1$	Child $Bn2$	Child $Bn3$	Child $Pr1$	Child $Pr2$	Child $Pr3$
5	45	55	7	9	35	11
12	28	48	24	40	12	60
13	53	73	25	41	37	61
Parent	Child $Bn1$	Child $Bn2$	Child $Bn3$	Child $Pr1$	Child $Pr2$	Child $Pr3$
7	91	105	9	13	63	15
24	60	88	40	84	16	112
25	109	137	41	85	65	113

Table 3: *Generating triples first from Parent* $(3, 4, 5)$*, then from* $(7, 24, 25)$*, by linear transformations: Symbol* $Bn \rightarrow Berggren, Pr \rightarrow Price$

In Table 3 above, child triples are being produced by applying the Berggren and Price transformations first on the parent $(5, 12, 13)$, and then on the parent $(7, 24, 25)$.

Firstly, it can be seen that by both set of transformations triples representing bigger triangular areas are produced from a given triple, not the other way round. Secondly, I would like to look at the consequences if I try to find out the parent of a given triple.

Berggren's second transformation matrix and its inverse are

$$T_{Bn2} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} \qquad \text{and} \qquad T_{Bn2}^{-1} = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ -2 & -2 & 3 \end{pmatrix}.$$

With this inverse matrix, I can find out the parent $(7, 24, 25)$ from the child $(105, 88, 137)$, and the parent $(5, 12, 13)$ from the child $(55, 48, 73)$.

It may be noted that the parent-child is always a unique and irreversible pair.

Thus going backwards, I have

$$T_{Bn2}^{-1}\begin{pmatrix}3\\4\\5\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}$$

which represents a "trivial" triple.

This shows that Berggren's transformations are based on $(3,4,5)$ as the "original" parent in the family of Pythagorean triples. This also implies that any triple can be decomposed as *a finite chain of linear transformations starting from* $(3,4,5)$, unlike our *infinitely long chain of compositions*.

Similarly, let us take Price's first transformation matrix and its inverse

$$T_{Pr1}=\begin{pmatrix}2&1&-1\\-2&2&2\\-2&1&3\end{pmatrix}\qquad\text{and}\qquad T_{Pr1}^{-1}=\frac{1}{4}\begin{pmatrix}2&-2&2\\1&2&-1\\1&-2&3\end{pmatrix}$$

by which, as in the case of Berggren transformations, we can establish unique pairs of parent-child relations.

Going backwards to $(3,4,5)$, I have

$$T_{Pr1}^{-1}\begin{pmatrix}3\\4\\5\end{pmatrix}=\frac{1}{2}\begin{pmatrix}4\\3\\5\end{pmatrix}$$

so the parent of triple $(3,4,5)$, according to the Price transformation, is the non-PPT $(8,6,10)$. The PPT equivalent to $(8,6,10)$ is $(4,3,5)$. But as the transformations are based on a being odd and b being even, this result with $a=4$, $b=3$ violates the primary condition of the formulation. This shows that Price's transformation cannot produce a parent of $(3,4,5)$ in a way that agrees with the condition of the formulations. It also implies, as in the case of Berggren, that any triple can be decomposed *only as a finite chain of linear transformations, starting from* $(3,4,5)$.

8.2 New relationships revealed by Composition and Decomposition

Compared with the invariable/irreversible parent-child relationships established by the linear transformations of Berggren and Price, the processes of Composition and Decomposition reveal a completely new kind of relationships between Pythagorean triples.

In Table 1, we saw that the triple $(15, 8, 17)$ is a parent of $(3, 4, 5)$. In Table 2, we see that $(3, 4, 5)$ in its turn can also be the parent of $(15, 8, 17)$ if $(3, 4, 5)$ is composed with its *amicable pairs*.

The phenomenon of infinitely many recurrences (discussed in *Section 4*) implies that any PPT can be generated through infinitely many different compositions. The topic of Decomposition very clearly shows that any PPT can generate any other if it finds its suitable *amicable pair*, which are 6 in number. Therefore, any PPT can be either the parent or child of any other; that is, all PPTs are mutually linked in *reversible parent/child relationships.*

Moreover, any triple can be expressed or broken up as a *finite* as well as an *infinitely long chain of compositions.*

Conclusion

With the help of six *Composition Laws* any two Pythagorean triples can be composed in six different ways to generate six other triples, which can be reduced to primitive triples according to *matrix equations* by which such compositions can be represented. Composing *ad infinitum* within this ever-increasing set of triples, all the infinitude of distinct primitive triples can be generated including infinitely many recurrences thereof. Though this implies that any primitive triple can be generated by infinitely many different compositions of triples, nevertheless six *Decomposition Laws*, including *matrix equations*, can be set up to find the triples with which any triple can be composed to generate a given triple. So any triple can be expressed as a *finite* as well as an *infinitely long chain of compositions. Composition/Decomposition* therefore reveals that any triple can generate any other if it finds its appropriate pair for composition. Thus no triple can be called more "original" than another in the family of Pythagorean triples; and all triples are interlinked in *reversible parent/child relationships*.

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