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On Sylvester's Sequence and some of its properties

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1 Introduction

Many of us recall first seeing sequences in elementary school, but it does not end there - sequences also commonly feature in research as they allow one to observe patterns in mathematical objects (numbers, matrices, etc.). Over the years, mathematicians have crossed great lengths in order to improve our understanding of some sequences, but there is far more out there that awaits to be discovered and understood, and Sylvester's Sequence is merely an example of the latter.

Sylvester's Sequence is named after the renowned English mathematician J.J. Sylvester who is said to have introduced it in 1880 by in [1], although some sources attribute the sequence to E. Lucas [2]. The sequence was used by Sylvester to study Egyptian fractions, which we shall later visit. Nearly a century and a half since its introduction, the sequence continues to be relevant as it is the focus of open conjectures, some of which we will note.

Definition 1. We define Sylvester's Sequence, denoted by $\{s_n\}_{n=0}^{\infty}$, by $s_0 = 1$ and the following recursive relationship:

$$s_{n+1} = 1 + \prod_{k=0}^{n} s_k \quad \text{for all} \quad n \geq 0 \; . \tag{1}$$

The first few terms of the sequence are 1, 2, 3, 7, 43, 1807, etc. Note that the index of multiplication in (1) can be altered to begin from $k = 1$ when $n \geq 1$ since $s_0 = 1$.

Property 2. Rather than defining s_n using all previous terms, s_{n+1} can also be written in terms of s_n alone, as follows for $n \geq 1$:

$$s_{n+1} = 1 + \prod_{k=1}^{n} s_k = 1 + s_n \prod_{k=1}^{n-1} s_k = 1 + s_n(s_n - 1) = s_n^2 - s_n + 1 \, . \tag{2}$$

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2 Algebraic treatment of Sylvester's Sequence

It can easily be verified that $\{s_n\}_{n=0}^{\infty}$ is a strictly increasing sequence of positive integers.

One of the most interesting facts regarding Sylvester's Sequence pertains to the sum of its reciprocals. Consider the partial sums of the sequence of Sylvester reciprocals, given by

$$u_n := \sum_{k=1}^{n} \frac{1}{s_k}\,.$$

Note that $u_1 = \frac{1}{2}$, $u_2 = \frac{5}{6}$, $u_3 = \frac{41}{42}$. The pattern, in which the denominators are one less than Sylvester numbers, inspires us to consider the following property.

Property 3. $u_n = \dfrac{s_{n+1} - 2}{s_{n+1} - 1}$.

Proof. We proceed by induction. The base case has been dealt with above, so we are left with the inductive hypothesis. Assume that $u_n = \dfrac{s_{n+1} - 2}{s_{n+1} - 1}$. Then by (2),

$$u_{n+1} = u_n + \frac{1}{s_{n+1}} = \frac{1}{s_{n+1}} + \frac{s_{n+1} - 2}{s_{n+1} - 1} = \frac{s_{n+1}^2 - s_{n+1} - 2}{s_{n+1}^2 - s_{n+1}} = \frac{s_{n+2} - 2}{s_{n+2} - 1}\,,$$

which concludes the inductive step. □

Theorem 4 (Adapted from Croatia National Olympiad, 2005 [3])**.**

$$\sum_{k=1}^{\infty} \frac{1}{s_k} = 1\,.$$

Proof. By Property 3,

$$\sum_{k=1}^{\infty} \frac{1}{s_k} = \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{s_{n+1} - 2}{s_{n+1} - 1} = 1\,.$$

□

Alternatively, we may prove convergence of the series by applying the Limit Ratio Test on this series. Knowing that $s_n \to \infty$ for $n \to \infty$:

$$\lim_{n\to\infty} \left| \frac{s_n}{s_{n+1}} \right| = \lim_{n\to\infty} \frac{s_n}{s_n^2 - s_n + 1} = 0 < 1\,,$$

and the series must thus converge. Yet, this only shows that it converges, without giving a sum.

We now shift our attention back to the idea of Egyptian fractions, which was behind the introduction of Sylvester's Sequence. Recall that Egyptian fractions are the sums of distinct unit fractions, i.e., reciprocals of distinct positive integers. For example, $\frac{10}{21}$ is an Egyptian fraction because one may write it as $\frac{1}{3} + \frac{1}{7}$. Note that Theorem 4 provides an affirmative answer to whether 1 can be written as the sum of a non-geometric, infinite series of unit fractions.

Definition 5. For all $x \in \mathbb{Q}$ with $0 < x \leq 1$, let $\sigma(x)$ be the smallest positive integer n such that x can be written as the sum n distinct unit fractions. Likewise, we define

$$h_n(x) = \begin{cases} 1, & \text{if } x \text{ is the sum of } n \text{ distinct unit fractions} \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

It is well-known that every rational number $0 < x \leq 1$, can be written as a sum of unit fractions, so $\sigma(x)$ is indeed well-defined. An example of the construction could be Fibonacci's Greedy Algorithm; see [4]. However, we focus on the case where $x = 1$. Inspired by Property 3 and Definition 2, we present the following result.

Theorem 6 (Related to Kellog's Problem [5])**.**

$$h_n(1) = 1 \qquad \textit{for all} \quad n \geq 3\,.$$

Proof. Note that, by Property 3, we can write

$$\sum_{k=1}^{n-1} \frac{1}{s_k} + \frac{1}{s_n - 1} = u_n + \frac{1}{s_n - 1} = \frac{s_n - 2}{s_n - 1} + \frac{1}{s_n - 1} = 1\,.$$

For these to be distinct, $s_n - 1$ ought to be strictly greater than s_{n-1}, which implies that $n \geq 3$. This therefore shows that 1 can be written as the sum of n unit fractions. □

Soundararajan [6] also used elementary techniques along with Muirhead's Inequality to show that, for any n, the closest approximation of 1 as the sum of reciprocals of integers $a_1, a_2, \ldots, a_n$, i.e., such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = 1 - \epsilon$$

for ϵ as small as possible, is obtained when $a_i = s_i$ for all $1 \leq i \leq n$.

The previous two facts are related to Erdős' Conjecture, which roughly states that for a class of sequences obeying a certain asymptotic condition[2], the sum of its reciprocals is rational only when the sequence has the form of Sylvester's Sequence.

Conjecture 7 (Erdős, 1980 [7])**.** *Consider a sequence of positive integers a_n such that:*

(1) $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n^2} = 1$

(2) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{s_k} \in \mathbb{Q}$

Then, there exists N such that, for all $n \geq N$,

$$a_{n+1} = a_n^2 - a_n + 1\,.$$

[2]A condition on an expression as it approaches infinity.

Of course, Sylvester's Sequence obeys Condition (1), as

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n^2} = \lim_{n\to\infty} \frac{a_n^2 - a_n + 1}{a_n^2} = 1\,,$$

and by Theorem 4, Condition (2) also holds.

Condition (1) is, in fact, quite important and deserves a discussion of its own. In general, asymptotically exponential sequences are characterized by $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = c$ for some $c \in \mathbb{R}$. However, when a sequence obeys Condition (1), it is said to asymptotically be "doubly-exponential". While the simplest exponential sequence would be of the form a^x, doubly exponential functions could be of the form a^{b^x}. Remarkably, Sylvester's Sequence happens to be an example of the latter. Golomb [8] used elementary techniques to show that Sylvester's Sequence obeys the formula

$$s_n = \left\lfloor B^{2^{n-1}} + \frac{1}{2} \right\rfloor$$

for $B \approx 1.5979102$. From a computational perspective, the issue with this closed-form formula[3] is that to calculate the exact value of s_n, the constant B must be known to some required degree of accuracy in its decimal digits. Golomb obtained this constant using infinite products, and other results have expressed B it as an infinite sum, but the same issue of obtaining it to the required number of decimals persists.

3 Number-theoretic properties of Sylvester's Sequence

We now shift gears to a number-theoretic approach of the sequence.

Property 8 (Co-primality of distinct terms). *For all $i \neq j$,*

$$\gcd(s_i, s_j) = 1\,.$$

Proof. Without loss of generality, suppose that $i > j$ and recall that $s_i = 1 + \prod_{k=1}^{i-1} s_k$. Assume that a prime p divides $\gcd(s_i, s_j)$. Then $p | s_j | \prod_{k=1}^{i-1} s_k$, and we conclude that $p|1$, a contradiction. Thus any two distinct terms of the sequence are relatively prime. □

This observation is so simple yet important to the study of Sylvester sequences.

Problem 9 (Mathematical Olympiad Summer Program, 1997). *Prove that the sequence $1, 11, 111, 1111, \ldots$ contains an infinite sub-sequence whose terms are pairwise relatively prime.*

Proof. Note that the sequence whose n^{th} term is the decimal number comprised of n 1's is given by $a_n = \frac{10^n - 1}{9}$. As shown in [9], $\gcd(x^m - 1, x^n - 1) = x^{\gcd(m,n)} - 1$. Applying this result, we would like to find m and n that $\gcd(a_m, a_n) = 1$. Thus,

$$1 = \gcd\Big(\frac{10^m - 1}{9}, \frac{10^n - 1}{9}\Big) = \frac{\gcd(10^m - 1, 10^n - 1)}{9} = \frac{10^{\gcd(m,n)} - 1}{9}$$

[3] i.e., a formula that expresses s_n as a function of n

or, equivalently, $10^{\gcd(m,n)} - 1 = 9$ which gives $\gcd(m, n) = 1$.

Hence, we are looking for an infinite number of indices which are pair-wise co-prime. Recalling the Sylvester Sequence, we see that it satisfies the aforementioned co-primality, and thus, the subsequence $\{a_{s_n}\}_{n=1}^{\infty}$ satisfies the problem's conditions. □

The co-primality of the successive terms can also be used to prove the existence of infinitely many primes, as in Euclid's famous proof.

A number of open conjectures on the number-theoretic properties of Sylvester's Sequence still exist. An integer a is defined *square-free* if no prime p satisfies $p^2|a$. It is conjectured that all numbers in Sylvester's Sequence are square-free but this result is far from proven. Calculations do indicate that this is true for all known Sylvester numbers; see [10]. However, some elementary results exist on the prime divisors of the Sylvester sequence. One may prove, through quadratic residues, that no primes of the form $6k - 1$ divide terms of the sequence.

Property 10 ([11]). *s_n is never divisible by a prime of the form $p = 6k - 1$.*

Proof. Assume that $p|s_n$ for some prime $p = 6k - 1$ and integers n and k. Recall that $s_n = s_{n-1}^2 - s_{n-1} + 1$. Thus, we have

$$s_{n-1}^2 - s_{n-1} + 1 \equiv s_n \equiv 0 \pmod{p}.$$

Multiplying by 4, and factoring the square term, we get

$$4s_{n-1}^2 - 4s_{n-1} + 4 \equiv 0 \pmod{p}$$

so

$$\Big(2s_{n-1} - 1\Big)^2 \equiv -3 \pmod{p}.$$

Thus, we get that

$$\left(\frac{-3}{p}\right) = 1 \tag{4}$$

where, for any odd prime q,

$$\left(\frac{a}{q}\right) = \begin{cases} 1, & \text{if there exists } x \text{ such that } x^2 \equiv a \pmod{q}; \\ -1, & \text{otherwise} \end{cases}$$

is the Legendre symbol. This symbol satisfies the following easily-proven properties

$$\left(\frac{a}{q}\right)\left(\frac{b}{q}\right) = \left(\frac{ab}{q}\right) \quad \text{and} \quad \left(\frac{a}{q}\right) = \left(\frac{a+q}{q}\right) \tag{5}$$

as well as less easily-proven properties such as

$$\left(\frac{-1}{q}\right) = (-1)^{\frac{q-1}{2}} \tag{6}$$

and Gauss' Law of Quadratic Reciprocity:

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} . \tag{7}$$

By (4) and (5),

$$1 = \left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) . \tag{8}$$

Moreover, since, $p \equiv -1 \pmod 6$, then $p \equiv 2 \pmod 3$ and so, by (5),

$$\left(\frac{p}{3}\right) = \left(\frac{2}{3}\right) = -1 .$$

Also, by (7),

$$\left(\frac{3}{p}\right)\left(\frac{p}{3}\right) = (-1)^{\frac{p-1}{2}\frac{3-1}{2}} = (-1)^{\frac{p-1}{2}} , \tag{9}$$

so

$$\left(\frac{3}{p}\right) = (-1)^{\frac{p+1}{2}} .$$

Therefore by (8),

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p+1}{2}} . \tag{10}$$

But this contradicts (6), so p cannot divide s_n. □

An amusing question to ask ourselves is whether any Sylvester numbers happen to be Fibonacci numbers too.

Theorem 11. *The only numbers which appear in both Sylvester's and Fibonacci's sequences are* $1, 2, 3$.

Proof. This is equivalent to searching for m, n such that $s_m = F_n$. Our main approach is to compare the residues of either sequence modulo some integer. For Sylvester's Sequence, we search for integers that, reduced $\pmod r$, are eventually constant; that is, we search for some N such that for all $m \geq N$, $s_{m+1} \equiv s_m^2 - s_m + 1 \equiv s_m \pmod r$. As for Fibonacci's sequence, the recursive nature of the sequence, and the subsequent Pisano periods reduced $\pmod r$, provide us with the tools required to solve this problem, as in [12]. The periodicity of the sequence $\pmod r$ is guaranteed because there are only r^2 possible consecutive values, so by the Pigeonhole Principle, there must be a pair that when reduced $\pmod r$ occurs twice. Since the Fibonacci sequence recursively defines a term based on the two previous terms, we have thus established the periodicity of the sequence $\pmod r$. We have also attached the Python code for the search for periods in the Addendum.

Assume $s_m = F_n$ and $m \geq 3$. We claim that $s_m \equiv 3 \pmod 4$. This is certainly true for $m = 3$. Since $s_{m+1} = s_m^2 - s_m + 1$, an inductive argument shows that this is true for all $m \geq 3$ since $s_m \equiv 3 \pmod 4$ implies that $s_{m+1} \equiv s_m^2 - s_m + 1 \equiv 7 \equiv 3 \pmod 4$.

The Fibonacci sequence, reduced $\pmod 4$, has a period of 6. Furthermore,

$$F_n \equiv s_m \equiv 3 \pmod 4 \quad \text{if and only if} \quad n \equiv 4 \pmod 6 . \tag{11}$$

Moreover, $s_m \equiv 7 \pmod 9$ for $m \geq 3$. This is clearly true for $m = 3$, as $s_m = 7$. Assume that this is true for s_m. Then, since $s_{m+1} = s_m^2 - s_m + 1$, we also get that $s_{m+1} \equiv s_m^2 - s_m + 1 \equiv 43 \equiv 7 \pmod 9$.

However, the Fibonacci sequence, reduced $\pmod 9$, has a period of 24. Using the attached code and results,

$$F_n \equiv s_m \equiv 7 \pmod 9 \text{ if and only if } \equiv 9, 15 \pmod{24}\,. \tag{12}$$

However, the deduction from (11) is that n must be even, while the deduction from (12) is that n is odd, a contradiction. Thus, if s_m is also a Fibonacci number, then $m \leq 2$. Checking the values, we see that $1, 2, 3$ happen to occur in both sequences. □

Alternatively, using the Chinese Remainder Theorem to combine the two equivalence $\pmod 4$ and $\pmod 9$, we could have also noted that 7 does not appear in the periodic residue cycle of the Fibonacci Sequence reduced $\pmod{36}$, and similarly obtained a contradiction. However, computationally speaking, the second proof is more efficient as it requires to calculate $6 + 24 = 30$ residues in total, while the period $\pmod{36}$ is of length 24.

4 Conclusion

Sylvester's Sequence is at the core of some interesting research and conjectures pertaining to number theory, as we have indicated. Although currently unproven, we also invite the reader to check that the first few terms of the sequence are indeed square-free. The reader is also invited to numerically verify that that Property 10 holds, and we highly suggest that the reader explores the sequence through its double-exponential nature and observing divisibility patterns throughout its terms. The *The On-Line Encyclopedia of Integer Sequences* [13] lists further information and references on Sylvester Sequences for the interested reader.

While we presented the application of Sylvester's Sequence to Egyptian fractions through an expository algebraic approach, this can also be viewed from a number-theoretic perspective. This is of major importance as the topic of Egyptian fractions dates centuries ago. The approach used in Theorem 11, using the periodic nature of residues modulo certain integers, could also be extended to other linear recurrences such as Lucas sequences.

Sequences, in general, continue to play a major role in both professional and recreational mathematics. Although they often offer us the ability to note and deduce patterns, we have seen, as is the case with Sylvester's Sequence, that some are not well understood - yet!

Addendum

Below, we have attached the code and its output for finding the period of the Fibonacci sequence $\pmod{9}$, and for figuring out when $F_n \equiv 7 \pmod{9}$. If the latter is true, then "compatible" is displayed beside the number.

Code:

```
def f(n):
    if n==0:
        return 0
    if n==1:
        return 1
    else:
        return f(n-1)+f(n-2)

def period(n):
    if f(n)%9==0:
        if f(n+1)%9==1:
            return True
        else:
            return False
    else:
        return False

n=2
while period(n) is False:
    if f(n)%9==7:
        print(n, "compatible")
        n+=1
    else:
        print(n, "failed")
        n+=1

print(n, 'is our period')
```

Output:

```
2 failed
3 failed
4 failed
5 failed
6 failed
7 failed
8 failed
9 compatible
10 failed
11 failed
12 failed
13 failed
14 failed
15 compatible
16 failed
17 failed
18 failed
19 failed
20 failed
21 failed
22 failed
23 failed
24 is our period
```

Thus, $F_{n+24} \equiv F_n \pmod{9}$, and $F_n \equiv 7 \pmod{9}$ if and only if $n \equiv 9, 15 \pmod{24}$. Replacing each 9 by 4 and each 7 by 3, respectively, we get the search for the period $\pmod{4}$.

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