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Solutions 1611–1620

Q1611 Suppose that the expression

$$\left(1+x^2+\frac{1}{x}\right)^{10}$$

is expanded and like terms collected. Find the coefficient of x^3.

SOLUTION We follow the ideas of Problem 1605. Noting that

$$x^3=(x^2)^m\left(\frac{1}{x}\right)^{2m-3},$$

we will obtain a term in x^3 by choosing m of the factors to supply an x^2 and $2m-3$ of the remaining $10-m$ factors to supply a $1/x$; all the other factors must supply a 1. The number of ways of doing this is

$$\binom{10}{m}\binom{10-m}{2m-3}.$$

To obtain a final result of x^3, clearly at least two factors of x^2 must be included; so $m\geq 2$. The total number of terms specified above is $3m-3$, which is at most 10; so $m\leq 4$. So the relevant values of m are $2,3,4$, and the coefficient of x^3 is

$$\binom{10}{2}\binom{8}{1}+\binom{10}{3}\binom{7}{3}+\binom{10}{4}\binom{6}{5}=360+4200+1260=5820\,.$$

Alternatively, if you prefer Arya Kondur's method of trinomial coefficients (see the previous issue of *Parabola*): the expansion of the trinomial is

$$\begin{aligned}\left(1+x^2+\frac{1}{x}\right)^{10}&=\sum_{i+j+k=10}\binom{10}{i,j,k}1^i(x^2)^j\left(\frac{1}{x}\right)^k\\&=\sum_{i+j+k=10}\binom{10}{i,j,k}x^{2j-k},\end{aligned}$$

and the triples which will give a term in x^3 are $(7,2,1)$, $(4,3,3)$ and $(1,4,5)$. So the coefficient of x^3 is

$$\binom{10}{7,2,1}+\binom{10}{4,3,3}+\binom{10}{1,4,5}=\frac{10!}{7!\,2!\,1!}+\frac{10!}{4!\,3!\,3!}+\frac{10!}{1!\,4!\,5!}=360+4200+1260=5820$$

as above.

Q1612 Find all positive integers x, y such that

$$\frac{x+y}{x^2-xy+y^2}=\frac{8}{73}\,.$$

SOLUTION The equation

$$8(x^2-xy+y^2)=73(x+y) \tag{$*$}$$

shows that $x+y$ is a multiple of 8; therefore $x-y=(x+y)-2y$ is the difference of two even numbers and hence is even. So we substitute

$$u=\frac{x+y}{8}\,,\quad v=\frac{x-y}{2}\,,$$

which is equivalent to

$$x=4u+v\,,\quad y=4u-v\,.$$

Note that u, v are integers, and u is positive. Substituting into $(*)$ after dividing both sides by 8, we get

$$(16u^2+8uv+v^2)-(16u^2-v^2)+(16u^2-8uv+v^2)=73u\,,$$

which simplifies to

$$3v^2=73u-16u^2\,.$$

Now $3v^2$ cannot be negative; so $73u-16u^2\geq 0$; as u is positive, this gives $0<u<\frac{73}{16}$; and u is an integer, so we have $u=1,2,3,4$. It is easy to test these four possibilities: we find that $u=1,2,4$ do not give integer values for v. The only solution is $u=3$, $v=\pm 5$ and hence

$$x=17\,,\ y=7$$

or *vice versa*.

Q1613 Prove that any given string of decimal digits occurs (consecutively and in the given order) among the digits of n^2 for some integer n.

SOLUTION Let $2m$ be an **even number** which contains the given digits consecutively and in order. (So if the last of the digits is even we can just take the given digits; if it is odd then we would take, for example, the given digits followed by a zero.) Let k be the number of digits in $2m$. Take

$$n=m10^k+1$$

and consider

$$n^2=m^2\,10^{2k}+2m\,10^k+1\,.$$

Now the first term here ends in $2k$ zeros. The second consists of the digits of $2m$ (there are k of these) followed by another k zeros. This is $2k$ digits altogether, so if we add the first two terms then the digits of $2m$ are added to zeros and do not change. And the final 1 is then added to the last of the k zeros, and does not affect the digits of $2m$. Therefore n^2 contains the digits of $2m$, which contain the given digits. In more detail, if the digits of $2m$ are $d_1d_2\cdots d_k$ and the digits of m^2 are $c_1c_2\cdots c_{2k}$, then we have

$$\begin{aligned} n^2 &= c_1c_2\cdots c_{2k}\overbrace{00\cdots000\cdots0}^{2k\text{ digits}} + \overbrace{d_1d_2\cdots d_k00\cdots0}^{2k\text{ digits}} + 1 \\ &= c_1c_2\cdots c_{2k}d_1d_2\cdots d_k00\cdots1 \end{aligned}$$

which contains the required digits.

Q1614 Let $x_1, x_2, \ldots, x_n$ be n different positive integers in increasing order, and suppose also that $x_n < 2x_1$. That is,

$$x_1 < x_2 < \cdots < x_n < 2x_1\,.$$

Prove that if p is a prime number, s is a non–negative integer and the product $x_1x_2\cdots x_n$ is a multiple of p^s, then the quotient is at least $n!$; that is,

$$x_1x_2\cdots x_n \geq p^s n!\,.$$

SOLUTION We shall use mathematical induction on the product $x_1x_2\cdots x_n$: that is, we prove the result is true when the product is 1 (this is known as the *basis case*); and we shall show that we can deduce the result for any specific value of the product, provided we know that it is true for all smaller values. This will show that the result is true in all cases. If you have not met *proof by mathematical induction* and you are a student, then please ask your teacher.

We begin by noting that the result is obvious when $n = 1$ (since x_1 is a multiple of p^s we have $x_1 \geq p^s$), and also for $s = 0$ (since $x_1x_2\cdots x_n$ is a product of n different positive integers, it is at least $1 \times 2 \times \cdots \times n = p^0 n!$). So we need no further proof in these cases.

If the product is 1 we have $x_1x_2\cdots x_n = 1$; since the positive integers x_k are all different, there must be just one of them. So $x_1 = 1$, $n = 1$, and we have already noted that the result is true in this case. Thus we have proved the basis of the induction.

Now consider a case where the product $x_1x_2\cdots x_n$ is greater than 1, and assume that we already know the result is true for all smaller products. The integers x_k satisfy

$$x_1 < x_2 < \cdots < x_n < 2x_1\,;$$

this implies

$$x_2 \geq x_1 + 1\,,\ \ x_3 \geq x_1 + 2\,, \ldots,\ \ x_n \geq x_1 + (n-1)\,,\ \ 2x_1 \geq x_1 + n$$

and so $x_1 \geq n$. Suppose that $x_1x_2\cdots x_n$ is a multiple of p^s, with $s > 0$. Then some or all of the x_k will be multiples of p. Denote those that are multiples of p by $py_1,\ py_2,\ \ldots,\ py_m$, in increasing order, and those that are not by $z_1, z_2, \ldots, z_{n-m}$ (there may in fact be none of the zs). Observe that

$$(py_1)(py_2)\ldots(py_m)\,z_1z_2\ldots z_{n-m} = x_1x_2\cdots x_n$$

because the n integers on the left hand side are exactly the same as the n integers on the right hand side, though possibly in a different order. Now consider the numbers $y_1, y_2, \ldots, y_m$. We have

$$y_1 y_2 \cdots y_m = \frac{x_1 x_2 \cdots x_n}{p^m z_1 z_2 \cdots z_{n-m}} < x_1 x_2 \cdots x_n \,;$$

so the product of the ys is smaller than the product of the xs, and we therefore know that the result under consideration is true for the ys. Moreover,

$$x_1 \le py_1 < py_2 < \cdots < py_m \le x_n < 2x_1 \le 2py_1$$

and so

$$y_1 < y_2 < \cdots < y_m < 2y_1 \,;$$

also p^{s-m} is a factor of $y_1 y_2 \cdots y_m$; and hence

$$y_1 y_2 \cdots y_m \ge p^{s-m} m! \,.$$

It follows that

$$x_1 x_2 \cdots x_n \ge p^s \, m! \, z_1 z_2 \cdots z_{n-m} \,.$$

Finally, the zs are all different, and every z_k satisfies $z_k \ge x_1 \ge n$; and $m \le n$; so the expression

$$m! \, z_1 z_2 \cdots z_{n-m} = (1)(2) \cdots (m) z_1 z_2 \cdots z_{n-m}$$

is a product of n different positive integers, and therefore

$$x_1 x_2 \cdots x_n \ge p^s n! \,.$$

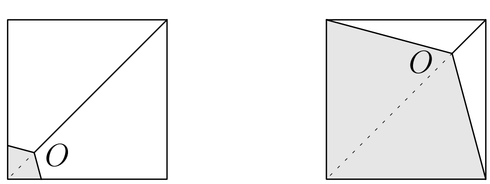
The proof is complete.

Q1615

(a) Show that it is possible to choose a point O inside a square and to draw three rays from O, all separated by equal angles, in such a way that the square is divided into three regions of equal area.

(b) Show that in (a), the point O cannot be the centre of the square.

SOLUTION Let the side of the square be 1, so that we are looking for three regions of area $\frac{1}{3}$ each. Place the point O on a diagonal of the square, very close to one corner, with one of the rays extending to the opposite corner (and the others spaced at angles of $120°$ as required) – see the first diagram.



Then the shaded area is very small, certainly less than $\frac{1}{3}$. Now gradually move the point O up the diagonal until it reaches the point shown in the second diagram: at this stage the shaded area is clearly bigger than $\frac{1}{2}$. So, somewhere in between, this area must be exactly $\frac{1}{3}$. The combined area of the other two regions is then $\frac{2}{3}$; and these two regions are clearly congruent, so they also have area $\frac{1}{3}$ each. This solves the problem.

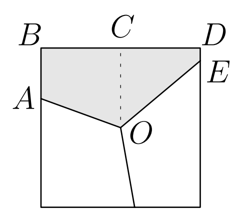
Comments.

- It is unnecessary to find the actual location of O in order to answer the question! However if you want to do so, this is not too hard and is left as an exercise. You should find that the distance from O to the bottom left corner of the square is

$$\frac{\sqrt{3+\sqrt{3}}}{3}\,.$$

- Another way to answer this question is to place O on the "mid–line" of the square, near the bottom, and then gradually move it vertically upwards. The argument is very similar to the one we have given.

For part (b), the three rays starting at O will meet the square in three points; so there must be a side of the square which does not meet any of the rays, except perhaps at a corner. If we draw this side as the top of the square and place O at the centre, it looks like this.



The shaded area is divided into two trapezoids by the dotted line; since $BC = CD = OC = \frac{1}{2}$, its area is

$$\Delta = \frac{1}{2}\frac{AB+OC}{2} + \frac{1}{2}\frac{OC+ED}{2} = \frac{AB+1+ED}{4}\,. \qquad (*)$$

Now let $\angle AOC = 60^\circ + \theta$; then $\angle EOC = 60^\circ - \theta$, and since A and E are on the vertical sides of the square we have $-15^\circ \le \theta \le 15^\circ$. Now

$$AB = \tfrac{1}{2} - \tfrac{1}{2}\tan(30^\circ - \theta)\,, \quad DE = \tfrac{1}{2} - \tfrac{1}{2}\tan(30^\circ + \theta)\,;$$

by using the formula

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

and the value $\tan 30^\circ = \frac{1}{\sqrt{3}}$, substituting back into $(*)$ and simplifying gives

$$\Delta = \frac{1}{4}\Bigl(2 - \sqrt{3}\,\frac{1+\tan^2\theta}{3-\tan^2\theta}\Bigr)\,.$$

Multiplying top and bottom of the fraction in this formula by $\cos^2\theta$ then using the identities $\cos^2\theta + \sin^2\theta = 1$ and $\cos^2\theta - \sin^2\theta = \cos 2\theta$, we have

$$\frac{1+\tan^2\theta}{3-\tan^2\theta} = \frac{\cos^2\theta+\sin^2\theta}{3\cos^2\theta-\sin^2\theta} = \frac{1}{1+2\cos 2\theta}$$

and so

$$\Delta = \frac{1}{4}\Big(2 - \frac{\sqrt{3}}{1+2\cos 2\theta}\Big)\,.$$

Now the minimum value of the area occurs for the minimum value of $\cos 2\theta$; this is when $\theta = \pm 15^\circ$, and the minimum area is

$$\Delta_{\min} = \frac{1}{4}\Big(2 - \frac{\sqrt{3}}{1+\sqrt{3}}\Big) = \frac{\sqrt{3}+1}{8}\,.$$

Finally, noting that $3\sqrt{3} = \sqrt{27} > \sqrt{25} = 5$, the area of the shaded region satisfies

$$\Delta \ge \Delta_{\min} = \frac{3\sqrt{3}+3}{24} > \frac{5+3}{24} = \frac{1}{3}\,.$$

Therefore the shaded area can never be $\frac{1}{3}$, and it is impossible to divide the square into three regions of area $\frac{1}{3}$ by means of three equally spaced rays meeting at the centre of the square.

NOW TRY Problem 1621.

Q1616 Fourteen circular counters of identical size are available; 9 of them are red and 5 are blue. In how many ways can they be arranged into a stack of 14 counters, if there cannot be more than 3 adjacent counters of the same colour?

SOLUTION Call a set of adjacent counters of the same colour a “stripe”. For example, the arrangement RRRBBRRBRBBRRR has four red stripes and three blue stripes. The number of red and the number of blue stripes must be the same, or must differ by 1. If the numbers are the same, either colour can come first and there are two possibilities; if they differ by 1, there is only one possibility. Now since there are nine red counters with at most three in each stripe, there must be at least 3 red stripes and at least 2 blue stripes; since there are five blue counters with at least one in each stripe, there are at most 5 blue stripes and at most 6 red stripes. At this stage, the numbers of possibilities for r red stripes and b blue stripes are as in the following table.

$b =$	2	3	4	5
$r = 3$	1	2	1	0
4	0	1	2	1
5	0	0	1	2
6	0	0	0	1

Now we must consider how many individual counters are in each stripe. For $r = 3$ red stripes, the only way to have a total of 9 red counters with a maximum of 3 in each stripe is to have $3+3+3$. For $r=4$ we could have $3+2+2+2$, or any re-ordering of these numbers: there are four possibilities. Alternatively we could have $3+3+2+1$, and there are 12 possibilities here (4 ways to choose the place to put the 1, then 3 ways to choose the place to put the 2). So for $r=4$ there are 16 possibilities altogether. For $r=5$ one of the possibilities is $3+2+2+1+1$, and the number of possibilities is 30 (5 ways to place the 3, then $\binom{4}{2}=6$ ways to place the 2s). The rest is left up to you: completing the calculations for $r=5$ you should find that the total number of possibilities is 45, and the total for $r=6$ is 50. Similarly, with five blue counters, the numbers of possibilities for $b=2,3,4,5$ stripes are $2,6,4,1$ respectively. So we can update the above table to show the total number of options in each case.

$b=$	2	3	4	5
$r=3$	$1\times1\times2$	$2\times1\times6$	$1\times1\times4$	0
4	0	$1\times16\times6$	$2\times16\times4$	$1\times16\times1$
5	0	0	$1\times45\times4$	$2\times45\times1$
6	0	0	0	$1\times50\times1$

Now all we need is to do the multiplications and add up the figures in the table, giving a final total of 578 possible arrangements.

Q1617 A **dyadic fraction** is a fraction in which the denominator is a power of 2, that is, a fraction

$$\frac{s}{2^n}$$

where s,n are integers and $n\geq 0$.

(a) Show that the sum, difference and product of two dyadic fractions is always a dyadic fraction.

(b) Find two dyadic fractions whose quotient is not a dyadic fraction.

Now let F be the set of all dyadic fractions,

$$F=\left\{\frac{s}{2^n}\,\middle|\, s,n \text{ are integers with } n\geq 0\right\}.$$

If a and b are specific numbers, then we write $aF+b$ for the set of all numbers that can be expressed as $ax+b$, where x is in F. That is,

$$aF+b=\{\,ax+b\mid x \text{ is in } F\,\}.$$

(c) Let a and b be fractions in F. Prove that $aF+b=F$ if and only if a is a power of 2, that is, $a=2^k$ for some integer k. (Note that k may be positive, negative or zero.)

Comment. In future issues we shall be presenting a series of problems about dyadic fractions, leading to questions which have been found important in a current mathematical research project. We hope that readers will be interested to see that even advanced contemporary mathematics sometimes relies on arguments which are accessible to school students.

SOLUTION Take two dyadic fractions x, y. We can write them as

$$x = \frac{s_1}{2^{n_1}}\,, \quad y = \frac{s_2}{2^{n_2}}$$

where s_1, n_1, s_2, n_2 are integers with $n_1, n_2 \geq 0$. Then

$$x + y = \frac{2^{n_2}s_1 + 2^{n_1}s_2}{2^{n_1+n_2}}\,,$$

and this is a dyadic fraction because the numerator is an integer and the denominator is a power of 2. For similar reasons,

$$x - y = \frac{2^{n_2}s_1 - 2^{n_1}s_2}{2^{n_1+n_2}} \quad \text{and} \quad xy = \frac{s_1 s_2}{2^{n_1+n_2}}$$

are dyadic fractions. This answers (a). There are many possible examples for (b). For instance, let $x = \frac{3}{2}$ and $y = \frac{5}{4}$. Then we can write

$$x = \frac{3}{2^1}\,, \quad y = \frac{5}{2^2}\,, \quad \frac{x}{y} = \frac{6}{5}\,;$$

so x and y are dyadic fractions, but $\frac{x}{y}$ is not since its denominator is not a power of 2.

To answer (c), let a and b be dyadic fractions. We have to prove firstly that if a is a power of 2, then every fraction in the set $aF + b$ is also in F, and conversely that every fraction in F is also in $aF + b$. We write

$$a = 2^k \quad \text{and} \quad b = \frac{s_2}{2^{n_2}}\,.$$

Any element of $aF + b$ can be expressed as $ax + b$, where x is a dyadic fraction; since a and b are also dyadic fractions, part (a) shows that $ax + b$ is a dyadic fraction and is therefore in the set F. Conversely, if x is in F then we have

$$x = \frac{s_1}{2^{n_1}} = \Big(\frac{s_1}{2^{n_1}} - \frac{s_2}{2^{n_2}}\Big) + b = a\Big(\frac{1}{2^k}\Big(\frac{s_1}{2^{n_1}} - \frac{s_2}{2^{n_2}}\Big)\Big) + b\,;$$

and the expression in brackets is a product and difference of dyadic fractions, so by (a) it is a dyadic fraction; hence x is in $aF + b$.

Finally, we have to show that if $aF + b = F$, where a, b are dyadic fractions, then a is a power of 2. So let

$$a = \frac{s}{2^n}\,.$$

Since b is a dyadic fraction, so is

$$\frac{1}{2^n} + b\,;$$

that is, this number is in F, and hence is also in $aF + b$ since this is the same set. Therefore

$$\frac{1}{2^n} + b = ax + b$$

for some x in F. But this gives

$$x = \frac{1}{s};$$

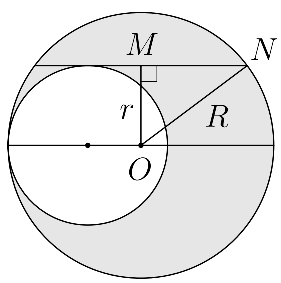
since x is a dyadic fraction, the denominator s is a power of 2; and therefore $a = s/2^n$ is a power of 2. This completes the proof.

Comment. In mathematical terminology, a set of numbers within which we can add, subtract and multiply in accordance with "sensible" rules is known as a *ring*. A set within which we can add, subtract, multiply and divide is called a *field*. So (a) shows, more or less, that the set F is a ring, while (b) shows that F is not a field. There is a great deal that can be said about these concepts: you may learn some of it if you study mathematics at university level.

NOW TRY Problem 1630.

Q1618 The points A, B, C are collinear, in that order. There is a circle on diameter AB and a circle on diameter AC. A chord of the larger circle is parallel to AC and tangent to the smaller circle; its length is $2x$. Find the area of the region lying between the two circles.

SOLUTION Let R be the radius of the larger circle and r the radius of the smaller. In the diagram, O is the centre of the larger circle and M is the foot of the perpendicular from O to the chord.



Since a chord of a circle is bisected by a perpendicular from the centre we have $MN = x$; and by Pythagoras' Theorem, $x^2 + r^2 = R^2$. Therefore the area of the shaded region is

$$\pi R^2 - \pi r^2 = \pi x^2 .$$

Q1619 A sequence of numbers $a_0, a_1, a_2, a_3, \ldots$ satisfies the equation

$$a_n = a_{n-1} + a_{n-2} + \lambda a_{n-1} a_{n-2} \qquad \text{for } n = 0, 1, 2, \ldots,$$

where λ is a fixed non–zero real number. Find a formula for a_n in terms of the first two values a_0, a_1.

SOLUTION Multiply both sides of the given equation by λ and add 1, giving

$$\lambda a_n + 1 = \lambda a_{n-1} + \lambda a_{n-2} + \lambda^2 a_{n-1} a_{n-2} + 1\,.$$

The right hand side now factorises and we have

$$\lambda a_n + 1 = (\lambda a_{n-1} + 1)(\lambda a_{n-2} + 1)\,. \qquad (*)$$

Writing $x = \lambda a_0 + 1$, $y = \lambda a_1 + 1$ and applying this equation, we find

$$\lambda a_2 + 1 = xy\,, \quad \lambda a_3 + 1 = xy^2\,, \quad \lambda a_4 + 1 = x^2y^3\,, \quad \lambda a_5 + 1 = x^3y^5$$

and so on. It is clear that $\lambda a_n + 1$ will be a power of x times a power of y, say

$$\lambda a_n + 1 = x^{s_n} y^{t_n}\,.$$

Substituting back into $(*)$ we have

$$x^{s_n} y^{t_n} = x^{s_{n-1}} y^{t_{n-1}} x^{s_{n-2}} y^{t_{n-2}} = x^{s_{n-1}+s_{n-2}} y^{t_{n-1}+t_{n-2}}\,.$$

Taking into account also the original definitions of x and y, we have

$$s_n = s_{n-1} + s_{n-2}\,,\ s_0 = 1\,, s_1 = 0\,, \quad t_n = t_{n-1} + t_{n-2}\,,\ t_0 = 0\,,\ t_1 = 1\,.$$

Therefore the values of t are the Fibonacci numbers, $t_n = F_n$. Since $s_1 = 0$ and $s_2 = 1$, the s values are the Fibonacci numbers "delayed" by one step, $s_n = F_{n-1}$. Therefore

$$\lambda a_n + 1 = x^{F_{n-1}} y^{F_n}\,.$$

Solving for a_n and substituting the original expressions for x and y gives the formula

$$a_n = \frac{x^{F_{n-1}} y^{F_n} - 1}{\lambda} = \frac{(\lambda a_0 + 1)^{F_{n-1}} (\lambda a_1 + 1)^{F_n} - 1}{\lambda}\,.$$

Q1620 There are m boxes, each containing some beads. A positive integer n is specified, with $n < m$. You are allowed to choose any n of the m boxes and then add one bead to each of the chosen boxes.

(a) Prove that if n and m have no common factor, then it is possible to perform this operation, more than once if necessary, in such a way that all boxes end up with the same number of beads.

(b) If n and m do have a common factor (greater than 1), find an initial distribution of beads such that it is impossible for the above operation, no matter how many times repeated, to result in all boxes containing the same number of beads.

SOLUTION We begin with an important result from the mathematical topic of *number theory*. If you are already familiar with this result, you can skip past it.

Theorem. Let m and n be positive integers which have no common factor (except 1). Then there is a positive integer k such that when kn is divided by m, the remainder is 1.

Proof. Consider the numbers $n, 2n, 3n, \ldots, mn$, and find the remainders when these numbers are divided by m. First note that all these remainders are different: for if $k_1 n$ and $k_2 n$ have the same remainder, then subtracting them cancels out the remainder, so that m is a factor of $(k_1 - k_2)n$. But since m and n have no common factor, m must be a factor of $k_1 - k_2$; and this is impossible as k_1, k_2 are numbers from 1 to m and their difference must be less than m.

So, if the m numbers $n, 2n, 3n, \ldots, mn$ are divided by m, they have m different remainders; and there are only m possible remainders, namely $0, 1, 2, \ldots, m-1$; so each remainder must occur once. In particular, one of the remainders must be 1, and this is what we wanted to prove.

Returning to the solution of our problem, let k be a number such that kn divided by m leaves remainder 1, say $kn = 1 + tm$. Place the m boxes in a circle, start with any box you like and go round and round the circle, placing one bead in each box until you have placed kn beads. This is within the rules of the problem, as it is the same as choosing n boxes and putting a bead in each, k times. And since there are m boxes, the result is that we will have gone around the circle t times, putting t beads into every box, and one extra in the first box. Thus we can increase the number of beads in any box we like by 1, relative to all the others. And if we do this a sufficient number of times, choosing as our first box any box which has less than the maximum number of beads, we shall eventually "level up" all the boxes to the same number of beads.

To answer (b), let m and n have a common factor $d > 1$. Start with one bead in one box and none in all the others. Then if we perform the allowed operation k times, the total number of beads in the boxes will be $kn + 1$; if the boxes then contain t beads each, the total number of beads will be tm; and these numbers are incompatible, as the latter is a multiple of d and the former isn't. Thus it is impossible to get the same number of beads in every box.