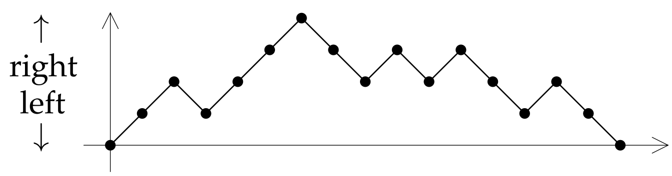
Parabola Volume 57, Issue 1 (2021)

Solutions 1631–1640

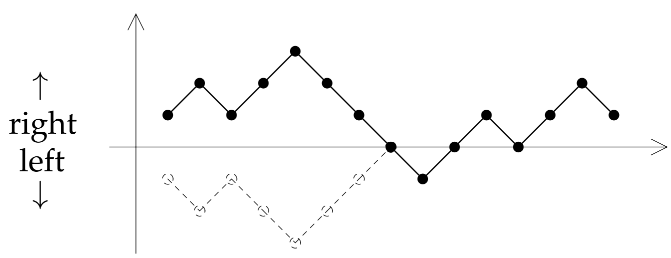
Q1631 As in Problem 1628, a path is labelled with the integers, and a person starting at 0 moves one step at a time left or right according to the flip of a fair coin. In the previous problem (whose solution appears in this issue), we found the probability that the person is back at the origin after $2n$ steps. Now see if you can find the probability that the person is back at the origin **for the first time** after $2n$ steps.

SOLUTION To get back to the origin for the first time after $2n$ steps, the person must always be on the same side of the origin. It is easy to see that the two sides are symmetrical; so we shall find the probability that the person is always on the positive side of the origin, returning to the origin at step $2n$, and then double this probability.

Draw a graph of location against step number: it might look, for example, like the following.



A path of the required type must start by moving from $(0, 0)$ to $(1, 1)$ and finish going from $(2n - 1, 1)$ to $(2n, 0)$. In between, it must go from $(1, 1)$ to $(2n - 1, 1)$ without ever touching the line $y = 0$. Paths from $(1, 1)$ to $(2n - 1, 1)$ must consist of $n - 1$ left steps and $n - 1$ right steps, so by the argument in Problem 1628, the number of paths is $C(2n - 2, n - 1)$ altogether; we need to subtract the number of such paths which **do** touch $y = 0$ at some point. Every path of this type looks something like the solid line in the following diagram.



If we take the path from $(1, 1)$ up to the first return to the origin and reflect it in the horizontal axis (dotted line in the diagram), then we obtain a path from $(1, -1)$ to $(2n - 1, 1)$, which must consist of n steps right and $n - 2$ steps left. The number of paths is $C(2n - 2, n)$, and so the number of paths we require is

$$\begin{aligned}\binom{2n-2}{n-1} - \binom{2n-2}{n} &= \frac{(2n-2)!}{(n-1)!\,(n-1)!} - \frac{(2n-2)!}{n!\,(n-2)!} \\ &= \frac{(2n)!}{n!\,n!}\left[\frac{n^2}{2n(2n-1)} - \frac{n(n-1)}{2n(2n-1)}\right]\end{aligned}$$

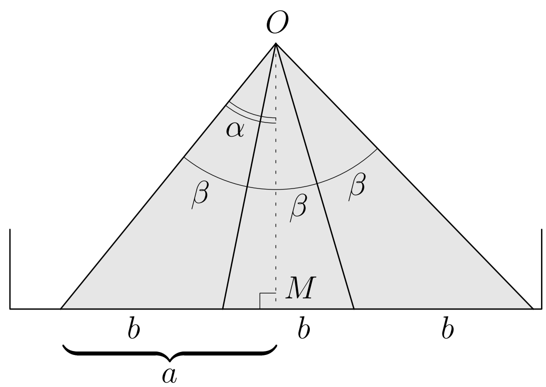
$$= \frac{1}{2(2n-1)}\binom{2n}{n}.$$

Dividing by the total number of paths consisting of $2n$ steps, and multiplying by 2 for the reason explained above, gives the probability of first return to the origin at step $2n$ as

$$f_{2n} = \frac{1}{2^{2n}(2n-1)}\binom{2n}{n}.$$

Q1632 Take a point O inside a square; from this point draw 13 rays, all spaced at equal angles. This will divide the square into 13 regions. Is it possible that all these regions have equal area?

SOLUTION Suppose that 13 equally spaced rays from the point O divide the square into 13 regions of equal area. As the rays meet the perimeter in 13 points, at least four of these points must lie on the same side of the square. So we have a situation like this,



where $\beta = \left(\frac{360}{13}\right)^\circ$. We shall measure lengths on the side of the square to the right of M, and angles anticlockwise from OM; so that in the example shown, the length a is negative, as is the angle α. Note, therefore, that

$$-90^\circ < \alpha < \alpha + 3\beta < 90^\circ .$$

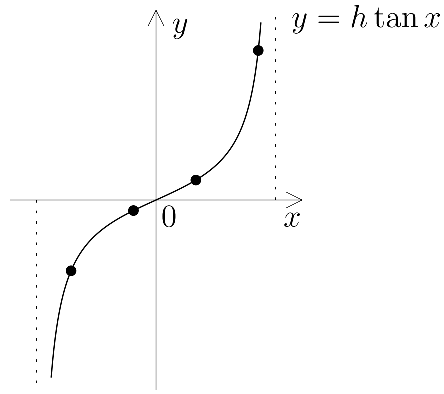
Now by assumption, the three shaded triangles have equal areas; and they also have equal altitudes $h = OM$; therefore, they have equal bases b, as marked in the diagram. From the four right–angled triangles in the diagram, we have

$$a = h\tan\alpha\ , \quad a+b = h\tan(\alpha+\beta)\ ,$$
$$a+2b = h\tan(\alpha+2\beta)\ , \quad a+3b = h\tan(\alpha+3\beta)\ ;$$

therefore the points

$$(\alpha,\, a)\ , \quad (\alpha+\beta,\, a+b)\ , \quad (\alpha+2\beta,\, a+2b)\ , \quad (\alpha+3\beta,\, a+3b)$$

all lie on the curve $y = h \tan x$ for $-90° \leq x < 90°$.



But these four points also satisfy the linear equation

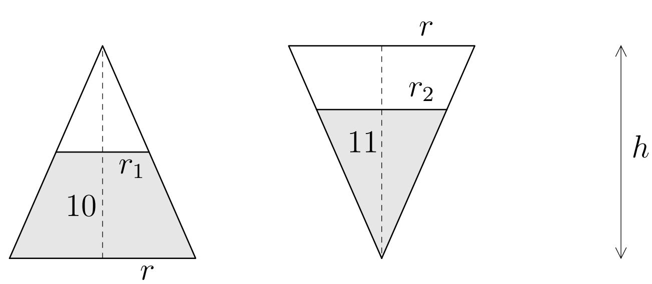
$$\frac{y - a}{b} = \frac{x - \alpha}{\beta}\,;$$

that is, the four points are collinear; and this is impossible, as the section of the tangent graph just shown clearly does not contain four collinear points. Thus, it is impossible for an arrangement of 13 rays as described to divide the square into regions of equal area.

Q1633 A right circular cone (with a closed base) is partially filled with water. The base of the cone is placed on a table and the depth of water in the cone is found to be 10cm. The cone is then inverted so that its vertex is on the table and its base is parallel to the table, and the depth of water is found to be 11cm. What height of empty space is there now above the water surface?

SOLUTION Thanks to UNSW student Yuan Yuan Wang for the following solution.

We begin this problem by drawing the two different scenarios; that is, base of the cone on table and vertex of the cone on table.



The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base of the cone and h is the height of the cone, so we apply this formula to find the volume of water in the cone within both scenarios. Let the radius of the water surface in the first scenario be r_1 and in the second scenario r_2, and finally let the height of the overall cone be h. Equating these two scenarios, we find that

$$\begin{aligned}\text{Volume of water} &= \frac{1}{3}\pi r^2 h - \frac{1}{3}\pi r_1^2(h-10)\\ &= \frac{1}{3}\pi(r^2 h - r_1^2(h-10))\\ &= \frac{1}{3}\pi r_2^2 \times 11\end{aligned}$$

(see figures above) and therefore

$$r^2 h - r_1^2(h-10) = 11 r_2^2\,. \tag{1}$$

To find the height of the empty space in the second scenario, we just need to calculate what $h-11$ is.

It's clear from a similar triangles argument that

$$\frac{r_1}{r} = \frac{h-10}{h} \quad \Rightarrow \quad r_1 = \frac{r(h-10)}{h} \tag{2}$$

$$\frac{r_2}{r} = \frac{11}{h} \quad \Rightarrow \quad r_2 = \frac{11r}{h}\,. \tag{3}$$

Hence, we can now substitute equations (2) and (3) into equation (1) to give

$$r^2 h - \frac{r^2(h-10)^2}{h^2}(h-10) = 11\,\frac{11^2 r^2}{h^2} \quad \Rightarrow \quad h - \frac{(h-10)^3}{h^2} = \frac{11^3}{h^2}\,.$$

Now, solving for h gives

$$\begin{aligned} h^3 - (h-10)^3 &= 11^3\\ h^3 - (h^3 - 30h^2 + 300h - 1000) &= 11^3\\ 30h^2 - 300h - 331 &= 0\end{aligned}$$

and so

$$h = \frac{150 \pm \sqrt{32430}}{30}\,.$$

Since $h > 0$, then

$$h = \frac{150 + \sqrt{32430}}{30} \approx 11.00277\cdots\ \text{cm}$$

and

$$h - 11 = 0.00277\cdots\ \text{cm} \approx 0.028\,\text{mm}\,.$$

Therefore, the height of empty space above the water surface is now around 0.028 mm.

Editor's comment: it may be surprising how small this is – approximately the width of one human hair!!

Q1634 Let n be a positive integer which is the sum of the squares of $2k$ positive integers, and suppose that no more than half of these squares are the same. Prove that $2n$ is also the sum of the squares of $2k$ positive integers. For example,

$$65 = 1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 5^2$$
$$130 = 1^2 + 2^2 + 3^2 + 4^2 + 6^2 + 8^2 \ .$$

Conversely, let $2m$ be an even number which is the sum of the squares of $2k$ positive integers. Suppose that no more than half of the odd squares are the same and no more than half of the even squares are the same. Prove that m is also the sum of the squares of $2k$ positive integers.

SOLUTION Suppose that n is the sum of $2k$ squares, no more than half of them the same. Then the squares can be arranged into k pairs $a_1^2, b_1^2, \ldots, a_k^2, b_k^2$, where all the a_j and b_j are positive integers and a_j is never equal to b_j: in fact, by symmetry we can arrange things so that $a_j > b_j$ in all cases. Then we have

$$n = a_1^2 + b_1^2 + \cdots + a_k^2 + b_k^2$$

and so

$$(a_1 + b_1)^2 + (a_1 - b_1)^2 + \cdots + (a_k + b_k)^2 + (a_k - b_k)^2$$
$$= 2a_1^2 + 2b_1^2 + \cdots + 2a_k^2 + 2b_k^2 = 2n \ ;$$

all of the terms $a_j \pm b_j$ are positive. Therefore, $2n$ is also the sum of squares of $2k$ positive integers.

Comment. The condition that no more than half the squares are the same is vital: if this is not the case, then the conclusion may not be true. For example, $7 = 1 + 1 + 1 + 4$ is the sum of four positive squares, but you can check by trial and error that 14 is not the sum of four positive squares.

For the second part of the question, suppose that $2m$ is the sum of $2k$ squares satisfying the stated condition. Since $2m$ is an even integer which is the sum of odd and even squares, there must be an even number (possibly none) of odd squares. Since no more than half of these odd squares are the same, we can pair them up in the same manner as above: $a_1^2, b_1^2, \ldots, a_l^2, b_l^2$, where all a_j and b_j are odd and $a_j > b_j$. Since there are an even number of odd squares in an even number ($2k$) of squares altogether, there must also be an even number of even squares and we can pair them up likewise: $a_{l+1}^2, b_{l+1}^2, \ldots, a_k^2, b_k^2$, where all these a_j and b_j are even and $a_j > b_j$. Now note that for every j, the integers a_j, b_j are both odd or both even; so the integers $a_j + b_j$ and $a_j - b_j$ are always even. This means that we can write a sum of squares of positive integers

$$\Big(\frac{a_1 + b_1}{2}\Big)^2 + \Big(\frac{a_1 - b_1}{2}\Big)^2 + \cdots + \Big(\frac{a_k + b_k}{2}\Big)^2 + \Big(\frac{a_k - b_k}{2}\Big)^2$$
$$= \frac{a_1^2 + b_1^2 + \cdots + a_k^2 + b_k^2}{2} = \frac{2m}{2} = m \ ,$$

and we have shown that m is the sum of $2k$ positive squares.

Q1635 Suppose that

$$x = \frac{1}{1\times 2} + \frac{1}{3\times 4} + \frac{1}{5\times 6} + \cdots + \frac{1}{2021\times 2022}$$

and

$$y = \frac{1}{1012\times 2022} + \frac{1}{1013\times 2021} + \frac{1}{1014\times 2020} + \cdots + \frac{1}{2022\times 1012}.$$

Find the value of x/y.

SOLUTION Firstly, for any k we have

$$\frac{1}{(2k-1)(2k)} = \frac{1}{2k-1} - \frac{1}{2k};$$

hence,

$$\begin{aligned} x &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2021} - \frac{1}{2022} \\ &= 1 + \frac{1}{2} - \frac{2}{2} + \frac{1}{3} + \frac{1}{4} - \frac{2}{4} + \cdots + \frac{1}{2021} + \frac{1}{2022} - \frac{2}{2022} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2021} + \frac{1}{2022} - 1 - \frac{1}{2} - \cdots - \frac{1}{1011} \\ &= \frac{1}{1012} + \frac{1}{1013} + \cdots + \frac{1}{2022}. \end{aligned}$$

Secondly, for any k we have

$$\frac{1}{(1011+k)(2023-k)} = \frac{1}{3034}\Big(\frac{1}{1011+k} + \frac{1}{2023-k}\Big);$$

and y is the sum of all these expressions for $k = 1, 2, 3, \ldots, 1011$. That is,

$$\begin{aligned} y &= \frac{1}{3034}\Big(\frac{1}{1012} + \frac{1}{2022} + \frac{1}{1013} + \frac{1}{2021} + \cdots + \frac{1}{2022} + \frac{1}{1012}\Big) \\ &= \frac{1}{3034}\Big(\frac{1}{1012} + \frac{1}{1013} + \cdots + \frac{1}{2022}\Big) \\ &\qquad + \frac{1}{3034}\Big(\frac{1}{2022} + \frac{1}{2021} + \cdots + \frac{1}{1012}\Big) \\ &= \frac{2}{3034}\Big(\frac{1}{1012} + \frac{1}{1013} + \cdots + \frac{1}{2022}\Big) \end{aligned}$$

and so $x/y = 1517$.

Q1636 In a right–angled triangle, a string with length equal to the hypotenuse is laid along the longer side, with the excess then going around the right angle and being laid along part of the shorter side. Find the maximum possible proportion of the shorter side covered by string, and the triangles which yield this maximum.

SOLUTION Let the sides of the right–angled triangle be a and b, with $a \le b$; so that the length of the string is $\sqrt{a^2+b^2}$. Then the length of string laid along the shorter side is $\sqrt{a^2+b^2}-b$, and the proportion covered is

$$p = \frac{\sqrt{a^2+b^2}-b}{a}\,.$$

Setting $x = b/a$, this can be written

$$p = \sqrt{1+x^2}-x = \frac{1}{\sqrt{1+x^2}+x}\,.$$

If x increases, the denominator of this fraction always increases and so p decreases; therefore, the maximum value of p occurs with the minimum value of x. And since $x = b/a$ with $b \ge a$, this gives a minimum value $x = 1$ and a maximum value $p = \sqrt{2}-1$. In this case we have $b = a$, so the triangles in question are isosceles right–angled triangles.

Q1637 Find the smallest multiple of 8642 which ends in the digits 2468.

SOLUTION To find a solution we can just repeat the method used to solve Problem 1625 (see solution in the previous issue of *Parabola*). Even easier, we can just take the answer to that problem, $4321 \times 9954 = 43011234$, and double both sides to get

$$8642 \times 9954 = 86022468\,.$$

However, this time we have not found the smallest solution. In this case, the associated equation (see previous solution for an explanation)

$$8642x = 10000y + 2468$$

can be simplified to

$$4321x = 5000y + 1234\,,$$

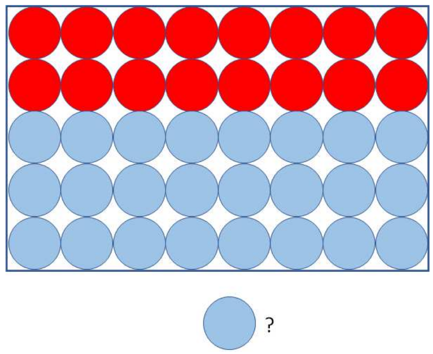
the general formula for x is

$$x = 9954 + 5000t \quad \text{where } t \text{ is an integer},$$

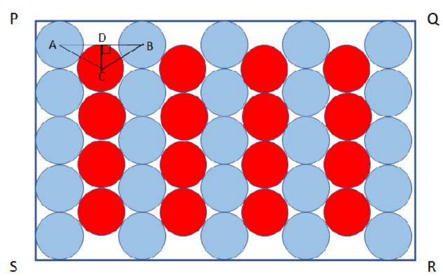
and we get the smallest positive solution by choosing $t = -1$. Therefore, the smallest multiple of 8642 which ends in the digits 2468 is

$$8642 \times 4954 = 42812468\,.$$

Q1638 A toy game designer devised a game consisting of 41 solid marbles. She noticed that a rectangular box for 40 marbles, packed as shown, is only $\pi/4 \approx 79\%$ full. How did she manage to fit one more marble into the box?



SOLUTION



Let the radius of each marble be 1, so that the dimensions of the box are 16 by 10. In the above diagram, we have $AB = 2AD = 2\sqrt{3}$. Thus the horizontal width of the new arrangement of marbles is $2 + 8\sqrt{3}$, which is less than 16; so the arrangement fits into the same box.

Q1639 What is the largest integer that **cannot** be expressed in the form $99a + 100b + 101c$, where a, b, c are non–negative integers?

SOLUTION We can place all possible values of $n = 99a + 100b + 101c$ into groups according to the value of $k = a+b+c$. For example, group 0 corresponds to $a+b+c = 0$, giving only one possibility: $(a, b, c) = (0, 0, 0)$ and $n = 0$. Another example: group 2 corresponds to $a + b + c = 2$, giving six possible triples

$$(a, b, c) = (2, 0, 0),\ (1, 1, 0),\ (0, 2, 0),\ (1, 0, 1),\ (0, 1, 1),\ (0, 0, 2)$$

and five different values

$$n = 198,\ 199,\ 200,\ 201,\ 202\,.$$

The smallest integer in group k is $n = 99k$, given by $(a, b, c) = (k, 0, 0)$; and the largest is $n = 101k$, given by $(a, b, c) = (0, 0, k)$. Furthermore, all intermediate integers are also found in group k. To see why this is true, imagine a basket containing k balls, all labelled 99; the total of all labels is $99k$. We can, one at a time, replace a "99 ball" by a "100 ball" or a "100 ball" by a "101 ball"; each time we do this, the total increases by 1; and so we obtain every possible integer up to the point at which all k balls are labelled 101 and the total is $101k$.

To find all possible values of n we need to take all groups collectively. Now, as we have just seen, group k consists of a sequence of consecutive integers; so does group $k + 1$; and there is a gap between these groups if and only if

$$101k + 1 < 99(k + 1) ,$$

which can be solved to give $k < 49$. Thus, group 48 goes up to $n = 4848$; group 49 starts at $n = 4851$; the intermediate values 4849 and 4850 are not in any group and are therefore not possible values of n. Moreover, there is no gap between group 49 and group 50; nor between group 50 and group 51; and so on. So there are no further missing values of n, and the largest integer which cannot be written in the form $99a + 100b + 101c$ is 4850.

Q1640 Recall that in Problems 1617 and 1630 we made the following definitions:

- F is the set of all dyadic fractions, that is, fractions in which the denominator is a power of 2; and for any set X, we write $aX + b$ for the set of all numbers which can be written $ax + b$, where x is in X.
- A set is *locally finite* if it has only finitely many elements in any finite interval of the real number line.
- A set X is called "important" if

  $$(2X) \oplus X \text{ and } (X + \tfrac{1}{2}) \oplus X \text{ are both locally finite,}$$

  where $(2X) \oplus X$ denotes the set of numbers which are in $2X$ or X but not both, and similarly for the second expression.

Now if a is any integer and m is a positive integer, we write $S_{a,m}$ for the set of all dyadic fractions $s/2^n$ such that s has remainder a when divided by m: that is,

$$S_{a,m} = \left\{ \frac{s}{2^n} \;\middle|\; n \geq 0 \text{ and } s \text{ divided by } m \text{ has remainder } a \right\} .$$

(a) Show that none of the sets $S_{a,m}$ is locally finite.

(b) Show that none of the sets $\overline{S_{a,m}}$ is locally finite.

(c) Show that if a is odd and m is a power of 2 (that is, $m = 2^l$ for some $l \geq 1$), then $S_{a,m}$ is important; and that no two of the sets

$$S_{3,4} \, , \; S_{5,8} \, , \; S_{9,16} \; \ldots , \; S_{2^k+1, 2^{k+1}}$$

have any elements in common.

Comment. The point of the question is this: we proved last issue that if either X or $\overline{X}$ is locally finite, then X is important. The present question shows that there are further types of important sets which we have not yet seen.

SOLUTION The argument for parts (a) and (b) is very similar to that in part (a) of Problem 1622. Consider any interval $p < x < q$, and let k be a positive integer such that

$$\frac{1}{2^k} < \frac{q-p}{m}\,.$$

Then for any $n \geq 0$, the interval

$$p2^{k+n} < x < q2^{k+n} \tag{$*$}$$

has length greater than $m2^n$. It can therefore be split into 2^n intervals of length m, each of which contains an integer s giving remainder a when divided by m. For each such s we have

$$\frac{s}{2^{k+n}} \text{ is in } S_{a,m} \quad \text{and} \quad p < \frac{s}{2^{k+n}} < q\,;$$

so $S_{a,m}$ has at least 2^n elements in the interval from p to q. And since n can be chosen as large as we like, $S_{a,m}$ contains infinitely many numbers in this interval. Therefore, $S_{a,m}$ is not locally finite.

For (b), once we have split the interval $(*)$ into 2^n intervals of length m, each interval contains an integer which *does not* give remainder a when divided by m. (In fact, each interval contains $m-1$ such integers – all those other than the s in part (a).) And by exactly the same argument, each such integer s satisfies

$$p < \frac{s}{2^{k+n}} < q \quad \text{and} \quad \frac{s}{2^{k+n}} \text{ is not in } S_{a,m}\,.$$

Thus $\overline{S_{a,m}}$ has at least 2^n elements in the interval from p to q, and hence is not locally finite.

Now let $X = S_{a,m}$ with a odd and $m = 2^l$, $l \geq 1$; we shall show that both $(2X) \oplus X$ and $(X + \frac{1}{2}) \oplus X$ are locally finite. First, if x is in X then $x = s/2^n$, where s has remainder a when divided by m; and we can write

$$x = 2\left(\frac{s}{2^{n+1}}\right),$$

which shows that x is also in $2X$. So it is impossible for any element to be in X but not $2X$, and therefore $(2X) \oplus X$ consists of dyadic fractions which are in $2X$ but not in X. The fractions in $2X$ are $x = 2(s/2^n)$, and the only way for this not to be in X is when $n = 0$. That is, $x = 2s$; so all elements of $(2X) \oplus X$ are even integers; and there are only finitely many of these in any interval $p < x < q$ on the real line. Thus, $(2X) \oplus X$ is locally finite.

Next we consider $(X + \frac{1}{2}) \oplus X$. Let $x = s/2^n$ with s odd, and suppose that $n \geq l + 1$. Then

$$x \text{ is in } X + \frac{1}{2} \quad \Leftrightarrow \quad \frac{s}{2^n} - \frac{1}{2} \text{ is in } X$$

$$
\begin{aligned}
\Leftrightarrow \quad & \frac{s-2^{n-1}}{2^n} \text{ is in } X \\
\Leftrightarrow \quad & s-2^{n-1} \text{ has remainder } a \text{ when divided by } 2^l \\
\Leftrightarrow \quad & s \text{ has remainder } a \text{ when divided by } 2^l \\
\Leftrightarrow \quad & x \text{ is in } X\,.
\end{aligned}
$$

This shows that dyadic fractions with denominator 2^{l+1} or greater must be in both $X+\frac{1}{2}$ and X, or in neither. Therefore, $(X+\frac{1}{2}) \oplus X$ contains only fractions with denominator 2^l or less; and there are only finitely many of these in any finite interval of the real line; so the set is locally finite.

To show that no two of the given sets have any elements in common: a dyadic fraction in two of the sets must be say

$$\frac{2^k+1+s2^{k+1}}{2^{n_1}} = \frac{2^l+1+t2^{l+1}}{2^{n_2}}\,,$$

where by symmetry we may assume $k < l$. Since the numerators are odd and the denominators are powers of 2, both fractions are in lowest terms; therefore the powers in the denominators are the same, and we have

$$2^k = 2^l + t2^{l+1} - s2^{k+1}\,.$$

But this is impossible since the right hand side is a multiple of 2^{k+1} and the left hand side is not. Therefore, no two sets in $(*)$ can share any elements.