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Problems 1661–1670

Parabola would like to thank Toyesh Prakash Sharma for contributing Problem 1670.

Q1661 Do the calculations for Problem 1657 without using calculus. Specifically,

(a) Find the gradient of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ at the point (p, q);

(b) Find the minimum value of $2a^4 - 2a^3\sqrt{a^2 - 4}$ for $a \geq 2$.

Q1662 This is a variation of Problem 1659. Now we have eight pairs of twins, and there are four activities, music, painting, reading and dancing, with four children to do each activity. Once again, each pair of twins is to do two separate activities. In how many ways can children be allocated to activities?

Q1663 As in Problem 1643 (*Parabola* Volume 57, Issue 1), a positive integer with k digits $d_0d_1 \cdots d_{k-1}$ in base 10 is called a *Geezer number* if the digits consist of exactly d_0 zeros, exactly d_1 ones, exactly d_2 twos and so on. Show that in a k–digit Geezer number, at most one of the digits $d_3, d_4, \ldots, d_{k-1}$ is non–zero.

Q1664 Let a, b, c, d be four prime numbers for which $5 < a < b < c < d < a + 10$. Prove that 60 is a factor of $a + b + c + d$ but 120 is not.

Q1665 Let $ABCD$ be a square, and let P and Q be the midpoints of AD and BC respectively. Suppose that PQR is the diameter of a circle passing through B and C and that $QR = 1$. Find the radius of the circle.

Q1666 The equation

$$\cos x + \sin y = \cos y + \sin x$$

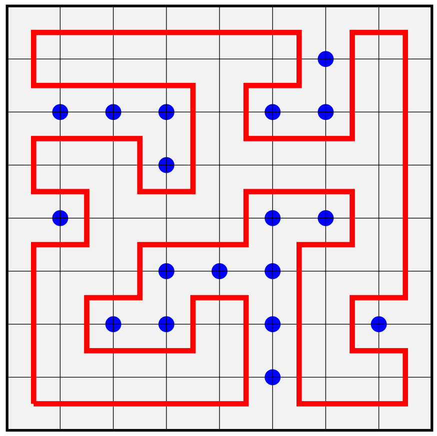
is obviously true when $\cos x = \cos y$ and $\sin x = \sin y$. Does it have any other solutions?

Q1667 A bag contains 8 balls, two each of 4 different colours. They are to be drawn from the bag in a random order and placed in a row.

(a) Find the probability that the row consists of pairs of the same colour, for example, red, red, black, black, blue, blue, white, white.

(b) Find the probability that the second half of the row has the same colours in the same order as the first half, for example, black, blue, white, red, black, blue, white, red.

Q1668 A closed path consists of lines from the centre of a square to the centre of an adjacent square on a $2n$ by $2n$ grid. The curve visits every square exactly once. An example is shown in the diagram.



There are a number of intersections of gridlines outside the path, shown as blue dots in the diagram. How many?

Q1669 Let OBC and ODA be right–angled isosceles triangles of the same size such that B and D are the right angles, the point B lies on OA and the point D lies on OC. Use this diagram to evaluate $\tan(\pi/8)$.

Q1670

(a) Prove the *weighted arithmetic–geometric mean inequality*: if a, b are positive numbers and $0 \leq p \leq 1$, then

$$a^p b^{1-p} \leq pa + (1-p)b\ .$$

(b) Use (a) to show that if $0 \leq x \leq \frac{\pi}{2}$ then

$$(\cos^2 x)^{\cos^2 x} + (\cos^2 x)^{\sin^2 x} + (\sin^2 x)^{\cos^2 x} + (\sin^2 x)^{\sin^2 x} \leq 3\ .$$