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An elementary proof that the regular polygon is the largest among polygons that are inscribed in a circle

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1 Introduction

In this article, we present an elementary proof of the following fact.

($*$) *A regular polygon has the largest area among all polygons inscribed in a circle.*

In this statement, it is assumed that all polygons have the same number of sides.

Most proofs for ($*$) require sophisticated knowledge of calculus. Jensen's inequality was used in [1], which is difficult for most high school students to prove. It is relatively easy to make a polygon of a larger area from an irregular polygon by making two neighbouring sides of the same length; this fact is sometimes presented as proof that the regular polygon has the largest area among polygons that are inscribed in a circle, but we need an advanced theorem to obtain rigid proof using this line of argument. Section 5 presents an outline of the proof presented in [2] because this proof demonstrates the difficulty of proving ($*$). We treat the case of polygons that have an even number of sides in Section 2, and the proof is elementary. This requires a high school freshman's knowledge, although the calculation is complicated, and we used the knowledge of geometric progressions.

We prove the case of ($*$) with n sides for an arbitrary natural number n in Section 3; this result generalises that of Section 2. In Section 3, we use a series of skilful techniques. The importance of Section 2 is that it only requires elementary knowledge of mathematics and is intuitively easy to understand.

The results in Sections 2 and 3 can be understood intuitively by many high-school students and can be introduced in the classroom. Most of the proofs were made by four Japanese high-school students, although a mathematician made them mathematically rigid.

Theorem 5 in Section 3 can be applied to other problems, and we present a problem for which this theorem is applied.

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2 The area of polygons with an even number of sides

Let n be a natural number. In this section, we treat only polygons with $2n$ sides. Here, we only use elementary geometry and some intuitive knowledge of the limits of sequences. Two types of operations are required for a finite number sequence.

Definition 1. We define two operations (a) and (b) on any finite sequence $a_1, a_2, \ldots, a_{2n}$:

(a) This operation creates the sequence

$$\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, \frac{a_3 + a_4}{2}, \frac{a_3 + a_4}{2}, \ldots, \frac{a_{2n-1} + a_{2n}}{2}, \frac{a_{2n-1} + a_{2n}}{2} .$$

(b) This operation creates the sequence

$$\frac{a_{2n} + a_1}{2}, \frac{a_2 + a_3}{2}, \frac{a_2 + a_3}{2}, \ldots, \frac{a_{2n-2} + a_{2n-1}}{2}, \frac{a_{2n-2} + a_{2n-1}}{2}, \frac{a_{2n} + a_1}{2} .$$

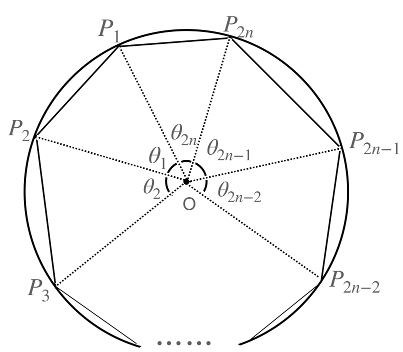

Figure 1: Polygon

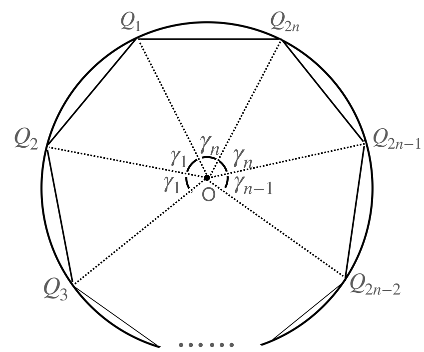

Figure 2: Polygon after (a) operation

Let O be the centre of the circles in Figure 1 and 2. We start with the polygon in Figure 1 and apply operation (a) to the central angles $\theta_1, \theta_2, \ldots, \theta_{2n}$. We thereby obtain the polygon in Figure 2 with the central angles $\gamma_1, \gamma_1, \gamma_2, \gamma_2, \ldots, \gamma_n, \gamma_n$. This procedure is used to prove in Theorem 2.

Theorem 2. *For each integer $n \geq 2$, the regular $2n$-sided polygon has the largest area among all $2n$-sided polygons inscribed in a given circle.*

Proof. Let $\theta_1, \theta_2, \ldots, \theta_{2n}$ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the angles given in Figures 1 and 2, and note that

$$\gamma_1 = \frac{\theta_1 + \theta_2}{2}, \ \gamma_2 = \frac{\theta_3 + \theta_4}{2}, \ \ldots, \ \gamma_i = \frac{\theta_{2i-1} + \theta_{2i}}{2}, \ \ldots, \ \gamma_n = \frac{\theta_{2n-1} + \theta_{2n}}{2}$$

and that

$$\gamma_1 + \gamma_2 + \cdots + \gamma_{n-1} + \gamma_n = \pi . \quad (1)$$

Since $\gamma_1 = \frac{\theta_1+\theta_2}{2}$, we have that $|P_1P_3| = |Q_1Q_3|$. Therefore, the triangular base of $\triangle Q_1Q_2Q_3$ is the same as that of $\triangle P_1P_2P_3$. The height of $\triangle Q_1Q_2Q_3$ is larger than $\triangle P_1P_2P_3$ because $|Q_1Q_2| = |Q_2Q_3|$. Then, the area of $\triangle Q_1Q_2Q_3$ is larger than that of $\triangle P_1P_2P_3$, and the area of polygon $Q_1Q_2Q_3O$ is larger than that of polygon $P_1P_2P_3O$. Similarly, polygon $Q_iQ_{i+1}Q_{i+2}O$ is larger than $P_iP_{i+1}P_{i+2}O$ for $i = 2, 3, \ldots, 2n-3$. We can also prove that polygon $Q_{2n-1}Q_{2n}Q_1O$ is larger than polygon $P_{2n-1}P_{2n}P_1O$.

By adding the areas of these polygons, we prove the following: the area of polygon $Q_1Q_2\cdots Q_nQ_1$ is larger than that of polygon $P_1P_2\cdots P_nP_1$.

Since we get

$$\gamma_1, \gamma_1, \ldots, \gamma_n, \gamma_n \tag{2}$$

by applying operation (a) to $\theta_1, \theta_2, \ldots, \theta_{2n}$, we obtain a larger polygon when we apply operation (a) to the central angles of the sides of the polygons. The same can be said for applying operation (b) to the central angles of the sides of the polygons.

Next, we apply operation (b) to (2) and obtain

$$\beta_{1,1} = \frac{\gamma_n + \gamma_1}{2},\ \beta_{1,2} = \frac{\gamma_1 + \gamma_2}{2},\ \beta_{1,2} = \frac{\gamma_1 + \gamma_2}{2},\ \ldots$$
$$\ldots,\ \beta_{1,n} = \frac{\gamma_{n-1} + \gamma_n}{2},\ \beta_{1,n} = \frac{\gamma_{n-1} + \gamma_n}{2},\ \beta_{1,1} = \frac{\gamma_n + \gamma_1}{2}. \tag{3}$$

Then, we use operation (a) for (3), and we get

$$\beta_{2,1} = \frac{\gamma_n + 2\gamma_1 + \gamma_2}{2^2},\ \beta_{2,1} = \frac{\gamma_n + 2\gamma_1 + \gamma_2}{2^2},$$
$$\beta_{2,2} = \frac{\gamma_1 + 2\gamma_2 + \gamma_3}{2^2},\ \beta_{2,2} = \frac{\gamma_1 + 2\gamma_2 + \gamma_3}{2^2},$$
$$\ddots$$
$$\beta_{2,n} = \frac{\gamma_{n-1} + 2\gamma_n + \gamma_1}{2^2},\ \beta_{2,n} = \frac{\gamma_{n-1} + 2\gamma_n + \gamma_1}{2^2}. \tag{4}$$

Then, we apply operation (b) to (4), and we get

$$\beta_{3,1} = \frac{\gamma_{n-1} + 3\gamma_n + 3\gamma_1 + \gamma_2}{2^3},$$
$$\beta_{3,2} = \frac{\gamma_n + 3\gamma_1 + 3\gamma_2 + \gamma_3}{2^3}, \qquad \beta_{3,2} = \frac{\gamma_n + 3\gamma_1 + 3\gamma_2 + \gamma_3}{2^3},$$
$$\ddots$$
$$\beta_{3,n} = \frac{\gamma_{n-2} + 3\gamma_{n-1} + 3\gamma_n + \gamma_1}{2^3},\ \beta_{3,n} = \frac{\gamma_{n-2} + 3\gamma_{n-1} + 3\gamma_n + \gamma_1}{2^3},$$
$$\beta_{3,1} = \frac{\gamma_{n-1} + 3\gamma_n + 3\gamma_1 + \gamma_2}{2^3}. \tag{5}$$

In (3), symbol $\beta_{1,1}$ has one γ_n and one γ_1, and, similarly, each $\beta_{1,j}$ has two different types of symbols for $j = 1, 2, \ldots, n$. In (4), each $\beta_{2,1}$ has one γ_n, two γ_1 and one γ_2, and each $\beta_{2,j}$ has four symbols for each $j = 1, 2, \ldots, n$, with three different types of symbols in each. In (5), each $\beta_{3,j}$ has eight symbols and four different types of symbols.

Before we continue to make $\beta_{j,i}$ by the (a) and (b) operations, we study the rule that determines the number of symbols and the number of different types of symbols.

Both $\beta_{2,n}$ and $\beta_{2,1}$ contain four symbols, so $\beta_{3,1} = \frac{\beta_{2,n}+\beta_{2,1}}{2}$ contains eight symbols. Although $\beta_{2,n}$ and $\beta_{2,1}$ have three different symbols, they share two symbols. Thus, $\beta_{3,1}$ has four different symbols. Similarly, $\beta_{2i-1,1}$ and $\beta_{2i,1}$ have 2^{2i-1} and 2^{2i} symbols, respectively. Furthermore, they have $2i$ and $2i+1$ different symbols, as we continue to create finite sequences

$$\beta_{2i-1,1}, \beta_{2i-1,2}, \beta_{2i-1,2}, \ldots, \beta_{2i-1,n}, \beta_{2i-1,n}, \beta_{2i-1,1}$$

and

$$\beta_{2i,1}, \beta_{2i,1}, \beta_{2i,2}, \beta_{2i,2}, \ldots, \beta_{2i,n}, \beta_{2i,n}$$

for $i = 1, 2, \ldots$ and $j = 1, 2, \ldots, n$. Therefore, $\beta_{n-1,j}$ has n different symbols.

Hence, by (1), we have for each $j = 1, 2, \ldots, n$ that

$$\beta_{n-1,j} \geq \frac{\gamma_1 + \cdots + \gamma_n}{2^{n-1}} = \frac{\pi}{2^{n-1}} \,. \tag{6}$$

Next, we apply operation (a) and operation (b) again. We are going to show that each angle converges to $\frac{\pi}{n}$, and hence we must prove that the difference between the angles converges to 0. When we compare the two angles, we can compare them after we reduce $\frac{\gamma_1+\cdots+\gamma_n}{2^{n-1}}$ from each angle.

First, we apply operation (b) to $\beta_{n-1,1}, \beta_{n-1,1}, \ldots, \beta_{n-1,n}, \beta_{n-1,n}$, and subtract $\frac{\gamma_1+\cdots+\gamma_n}{2^{n-1}}$ from each angle. Then, the differences

$$\begin{aligned}
\gamma_1' &= \frac{\beta_{n-1,n} + \beta_{n-1,1}}{2} - \frac{\gamma_1 + \cdots + \gamma_n}{2^{n-1}}, \\
\gamma_2' &= \frac{\beta_{n-1,1} + \beta_{n-1,2}}{2} - \frac{\gamma_1 + \cdots + \gamma_n}{2^{n-1}}, \\
&\vdots \\
\gamma_n' &= \frac{\beta_{n-1,n-1} + \beta_{n-1,n}}{2} - \frac{\gamma_1 + \cdots + \gamma_n}{2^{n-1}},
\end{aligned}$$

satisfy

$$\gamma_1' + \cdots + \gamma_n' = (\gamma_1 + \cdots + \gamma_n)\left(1 - \frac{n}{2^{n-1}}\right) .$$

By using a procedure similar to that used in (2), (3), (4), (5), and (6) we get numbers $\beta'_{i,j}$ satsifying

$$\beta'_{n-1,j} \geq \frac{\gamma_1' + \cdots + \gamma_n'}{2^{n-1}} \,.$$

If we subtract $\frac{\gamma_1'+\cdots+\gamma_n'}{2^{n-1}}$ from each angle and perform a similar task, we obtain $\gamma_1'', \cdots, \gamma_n''$ such that

$$\gamma_1'' + \cdots + \gamma_n'' = (\gamma_1' + \cdots + \gamma_n')\left(1 - \frac{n}{2^{n-1}}\right) = (\gamma_1 + \cdots + \gamma_n)\left(1 - \frac{n}{2^{n-1}}\right)^2 .$$

If we repeat this procedure m times, then the sum of the angles $\gamma_1^{(m)}, \dots, \gamma_n^{(m)}$ satisfies

$$\gamma_1^{(m)} + \cdots + \gamma_n^{(m)} = (\gamma_1 + \cdots + \gamma_n)\Big(1 - \frac{n}{2^{n-1}}\Big)^m ,$$

and the sum converges to 0 as $m \to \infty$. Therefore, the difference between the angles tends to 0 as we repeatedly apply the two operations to the central angles.

We have proved that by applying the operations (a) and (b) repeatedly, each central angle converges to $\frac{\pi}{n}$. Hence, the area of the polygon increases as it becomes closer to the regular polygon.

This indicates that the area of any $2n$-sided polygon inscribed in a circle is equal to or less than that of the regular $2n$-sided polygon inscribed in the same circle. Therefore, the regular polygon has the largest area among the polygons inscribed in a circle. □

Remark 3. In the proof of Theorem 2, we should use the Binomial Theorem and mathematical induction to make the proof mathematically rigid; however, it can be understood without these.

3 Polygons with arbitrarily many sides

In this section, we prove $(*)$ for n-sided polygons where $n \geq 3$ is an arbitrary natural number. Here, we use more knowledge of mathematics than in the previous section. We need mathematical induction and calculations related to the limit of sequences. The calculations used in the proof may appear very complicated; however, if explained intuitively, we assume that high school students will understand these ideas.

Definition 4. We define the following operation on any sequence of real numbers $b_1, b_2, \dots, b_n$ and any integer i with $1 \leq i < n$:

$(c)_i$ This operation creates the sequence

$$b_1, b_2, \dots, b_{i-1}, \frac{b_i + b_{i+1}}{2}, \frac{b_i + b_{i+1}}{2}, b_{i+2}, \dots, b_n .$$

Theorem 5. *Let $a_1, a_2, \dots, a_n$ be a sequence of real numbers and let $s = a_1 + a_2 + \cdots + a_n$. By repeatedly applying the operations $(c)_i$ for all $i = 1, 2, \dots, n$, we obtain a sequence*

$$s/n + \delta_1, s/n + \delta_2, \dots, s/n + \delta_n , \tag{7}$$

for real numbers $\delta_1, \delta_2, \dots, \delta_n$, where $|\delta_1|, |\delta_2|, \dots, |\delta_n|$ can be as small as desired. In other words, each number in the resulting sequences converges to s/n.

Proof. We prove this theorem through mathematical induction. If $n = 2$, then operation $(c)_1$ applied to a_1, a_2 yields $\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2}$. Therefore, (7) is valid for $n = 2$. Note that $\delta_1 = \delta_2 = 0$ in this case.

We assume that this theorem is valid for $n - 1$ and prove the case for n, where $n \geq 3$. We start with $a_1, a_2, \dots, a_n$ and apply the operations $(c)_i$ sufficiently many times to $a_1, a_2, \ldots, a_{n-1}$ to get a sequence $d_k+\delta_1, d_k+\delta_2, \ldots, d_k+\delta_{n-1}$ for real numbers $d_k, \delta_1, \delta_2, \ldots, \delta_{n-1}$, where $|\delta_1|, |\delta_2|, \ldots, |\delta_{n-1}|$ can be as small as desired. The remaining (last) number is $s-(n-1)d_k-(\delta_1+\cdots+\delta_{n-1})$. Therefore, we have the following sequence:

$$d_k+\delta_1, d_k+\delta_2, \ldots, d_k+\delta_{n-1}, s-(n-1)d_k-(\delta_1+\cdots+\delta_{n-1})\,. \tag{8}$$

After applying the operations $(c)_i$ to

$$d_k+\delta_2, d_k+\delta_3, \ldots, d_k+\delta_{n-1}, s-(n-1)d_k-(\delta_1+\cdots+\delta_{n-1})$$

sufficiently times, we obtain a sequence $d_{k+1}+\delta'_2, d_{k+1}+\delta'_3, \ldots, d_{k+1}+\delta'_n$ for real numbers d_{k+1}, $\delta'_2, \delta'_3, \ldots, \delta'_n$, where $|\delta'_2|, |\delta'_3|, \ldots, |\delta'_n|$ can be as small as desired. Then, the remaining (first) number can be described as $s-(n-1)d_{k+1}-(\delta'_2+\delta'_3+\cdots+\delta'_n)$, and we obtain a sequence of numbers

$$s-(n-1)d_{k+1}-(\delta'_2+\delta'_3+\cdots+\delta'_n), d_{k+1}+\delta'_2, d_{k+1}+\delta'_3, \ldots, d_{k+1}+\delta'_n\,. \tag{9}$$

We did not alter the value of $d_k+\delta_1$, the first element in (8), when we applied the operations to the last $n-1$ numbers; hence, $d_k+\delta_1$ is equal to the first element in (9):

$$d_k+\delta_1 = s-(n-1)d_{k+1}-(\delta'_2+\delta'_3+\cdots+\delta'_n)\,.$$

Therefore,

$$d_{k+1} = \frac{s-d_k-(\delta_1+\delta'_2+\delta'_3+\cdots+\delta'_n)}{n-1}\,,$$

so

$$d_{k+1}-\frac{s}{n} = -\frac{d_k-\frac{s}{n}}{n-1}-\frac{\delta_1+\delta'_2+\delta'_3+\cdots+\delta'_n}{n-1}\,.$$

If $|\delta_1+\delta'_2+\delta'_3+\cdots+\delta'_n|$ is sufficiently small, then

$$\left|d_{k+1}-\frac{s}{n}\right| \le \left|\frac{d_k-\frac{s}{n}}{n-1}\right| + \frac{1}{(n-1)^k}\,.$$

Therefore for each m,

$$\begin{aligned}\left|d_m-\frac{s}{n}\right| &\le \frac{1}{(n-1)}\left|d_{m-1}-\frac{s}{n}\right| + \frac{1}{(n-1)^{m-1}}\\ &\le \frac{1}{(n-1)}\left(\frac{1}{(n-1)}\left|d_{m-2}-\frac{s}{n}\right| + \frac{1}{(n-1)^{m-2}}\right) + \frac{1}{(n-1)^{m-1}}\\ &\;\;\vdots\\ &\le \frac{1}{(n-1)^{m-1}}\left|d_1-\frac{s}{n}\right| + \frac{m-1}{(n-1)^{m-1}}\,.\end{aligned}$$

Therefore, $d_m \to \frac{s}{n}$ when $m \to \infty$. The proof follows by induction. □

Remark 6. In the proof of Theorem 5, we use the fact that $\frac{m-1}{(n-1)^{m-1}}$ tends to zero when m increases. This is well-known but some high school students might not know this.

Theorem 7. *For each integer $n \geq 3$, the regular n-sided polygon has the largest area among all n-sided polygons inscribed in a given circle.*

Proof. Let $a_1, a_2, \ldots, a_n$ be the central angles of the polygon. By Theorem 5, the polygon's area grows closer to that of the regular polygon as we apply the operations $(c)_i$. We explained that the area always increases during the process in Theorem 2. Thus, the regular polygon has the largest area among the polygons inscribed in a circle. □

4 A problem of containers with different water levels

Theorem 5 can be applied to the following problem. We have a group of water containers in Figure 3 and an operation that makes the water levels of two neighbouring containers equal. An example of this operation is shown in Figures 3, 4 and 5. In Figure 4, we remove the slit between the 6th container and the 7th container. Then, the water levels of the two containers will be the same. Subsequently, we put the slit back, as shown in Figure 5. As we can see, this operation is mathematically the same as that used in Theorem 5. By Theorem 5, we can make the levels of the containers close to the average of the levels in Figure 3 by repeatedly applying the operation.

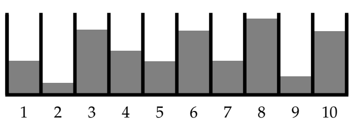

Figure 3: Water Levels 1

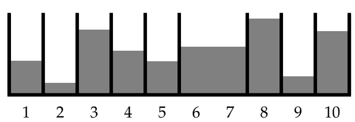

Figure 4: Water Levels 2

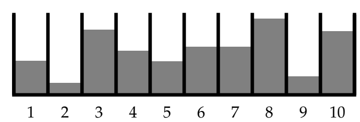

Figure 5: Water Levels 3

5 Appendix

Here, we outline proof, presented in [2], that the regular n-sided polygon is the largest among n-sided polygons that are inscribed in a circle. The proof is short and nice but depends on advanced-level calculus, unlike the proofs above in this article.

Proof. Let D be the set of vectors in R^n that describe the angles subtended by the sides of a non-self-intersecting cyclic polygon (which is allowed to have sides of length 0). Then D is clearly a closed bounded set and hence compact. Let $f(v)$ be the area of polygon P described by vector $v \in D$. Then, $f(v)$ is continuous within D. An advanced-level calculus theorem states that each continuous function attains a maximum on a compact set. Therefore, $f(v)$ attains a maximum value on D. Suppose that $f(a)$ is the maximum value for $a \in D$ and let R be the polygon described by a. If R is not regular, then R has two sides, AB and BC of unequal length. Move B on the arc to make AB and BC equal in length. It can be proved that the area of the new polygon thereby created is larger than R, a contradiction. Therefore, R is regular. □

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