Parabola Volume 4, Issue 1 (1967)

SOLUTIONS TO PROBLEMS IN PARABOLA, VOL.3, NO.3.

JUNIOR

J71 Find a four digit number which on division by 149 leaves a remainder of 17 and on division by 148 leaves a remainder of 29.

Answer: Let x represent the required four digit number, q_1 the quotient obtained when x is divided by 149 , and q_2 the quotient when the division is 148. Thus

$x = 149\ q_1 + 17$ and $x = 148\ q_2 + 29.$

Since x is a four digit number $7 \leq q_1, q_2 < 70$.
Subtracting the above equations gives

$149\ q_1 - 148\ q_2 = 12$

$148\ (q_1 - q_2) + q_1 = 12.$

Hence, since $q_1 - q_2$ is an integer its value must be zero, which gives

$q_1 = q_2 = 12$, and $x = 149 \times 12 + 17 = 1805.$

Correct: P.Crane, D.Thomson (Port Hacking H.S.) P.Berkemeier, E.Ketesz,,T.Ratcliffe, (St Aloysius Coll. A.Oberlander, E.Takacs,(Oak Flats H.S.) W.Bundschuh, (W.Wallsend H.S.) J.Irving, (Wiley Pk G.H.S.) A. Scholes, (Bathurst H.S.).

....

J72 Find the sum of the coefficients of the polynomial obtained after expanding and collecting the terms of the product $(1-5x + 5x^3)^{100} (1-2x + x^4)^{10}$.

Answer: The sum $a_n + a_{n-1} + \ldots + a_o$ of the coefficients of the polynomial $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_o$ is simply the value assumed by the polynomial when 1 is substituted for x. The value of $(1-5x+5x^3)^{100}(1-2x+x^4)^{10}$ when x = 1 is $(1-5+5)^{100}(1-2+1)^{10} = 1^{100}\ 0^{10} = 0.$

Correct answers received from: M.Waterworth (Fairfield B.H.S.).

OPEN

073 Prove that if the positive integers a_1, a_2 and $a_2 - a_1$ are all relatively prime to n, the cyclic Latin squares on n symbols con -structed from the first row by moving the symbols a_1 (or a_2) squares to the left to obtain each succeeding row, are orthogonal.

Answer: The Latin squares are

$$\begin{array}{ccccc}
1 & 2 & 3 & \ldots & n \\
a_1+1 & a_1+2 & a_1+3 & \ldots & a_1+n \\
2a_1+1 & 2a_1+2 & 2a_1+3 & \ldots & 2a_1+n \\
 & & \ldots\ldots\ldots\ldots & & \\
(n-1)a_1+1 & (n-1)a_1+2 & (n-1)a_1+3 & \ldots & (n-1)a_1+n
\end{array}$$

where $\kappa a_1 + \lambda$ is to be interpreted as x such that $0 < x \leq n$ and $x \equiv \kappa a_1 + \lambda \pmod{n}$, and same Latin square with a_1 replaced by a_2. Each number occurs exactly once in each row because the rows consist of the numbers $1,2,3,\ldots,n$ arranged in a certain order. Also no number is repeated in the columns, since if $\kappa_1 a_1 + \lambda \equiv \kappa_2 a_1 + \lambda \pmod{n}$, n divides $(\kappa_1 - \kappa_2)a_1$, but this is impossible since a_1 does not contain any of the prime factors of n and $\kappa_1 - \kappa_2$ does not contain them all, because $0 < |\kappa_1 - \kappa_2| < n$. The two squares are therefore Latin squares. Suppose the Latin squares are not orthogonal. Then there exist $\kappa_1, \kappa_2, \lambda_1, \lambda_2$ such that

$$\kappa_1 a_1 + \lambda_1 \equiv \kappa_2 a_1 + \lambda_2 \pmod{n}$$

and $$\kappa_1 a_2 + \lambda_1 \equiv \kappa_2 a_2 + \lambda_2 \pmod{n}.$$

From these congruences, $(\kappa_1-\kappa_2)(a_1-a_2) \equiv 0 \pmod{n}$, that is n divides $(\kappa_1-\kappa_2)(a_1-a_2)$, which is impossible since a_2-a_1 does not contain any of the prime factors of n and $0 < |\kappa_1-\kappa_2| < n$. The Latin squares are therefore orthogonal.

P. O'Sullivan.
(Randwick H.S.)

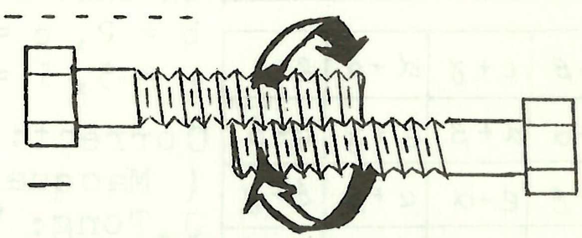
Correct answers received also from:

J. Tong (Waverley College)

P. Jablon (Telopea Pk. H.S.)

- - - - - - - - - - - - - - - -

<u>Bolts</u>



The diagram shows two identical bolts with their threads intermeshed. The bolts are moved around one another in the direction shown by the arrows, neither bolt being allowed to rotate on its axis. (If you twiddle your thumbs you will understand what is meant).

Do the heads of the bolts (a) move apart (b) stay the same distance apart or (c) get closer together?

(Answer page 24.).

074 In the array shown (which, if the plus signs are omitted is exactly the Graeco-Latin square of Fig.2A, p. 2) putting $a = 1$, $b = 3$, $c = 2$, $\alpha = 3$, $\beta = 0$, $\gamma = 6$ gives the familiar magic square using the first nine positive integers whose rows, columns and diagonals all sum to 15. Construct magic squares using the first 16 positive integers, and the first 25 positive integers, with the help of Graeco-Latin squares of order 4 and 5 respectively. (This problem explains why the title of Euler's article mentioned magic squares.)

$a+\alpha$	$b+\beta$	$c+\gamma$
$b+\gamma$	$c+\alpha$	$a+\beta$
$c+\beta$	$a+\gamma$	$b+\alpha$

Answer: In this G-L square putting $a = 2$, $b = 1$, $c = 3$, $d = 4$, $\alpha = 0$, $\beta = 12$, $\gamma = 8$, $\delta = 4$ gives a magic square using the first sixteen positive integers. You will observe that the values given to the symbols may be interchanged in obvious ways. In order that the main diagonals sum to the same value as the rows and columns it is merely necessary to have $a + c = b + d$ and $\alpha + \beta = \gamma + \delta$.

$b+\beta$	$c+\gamma$	$d+\delta$	$a+\alpha$
$a+\delta$	$d+\alpha$	$c+\beta$	$b+\gamma$
$d+\gamma$	$a+\beta$	$b+\alpha$	$c+\delta$
$c+\alpha$	$b+\delta$	$a+\gamma$	$d+\beta$

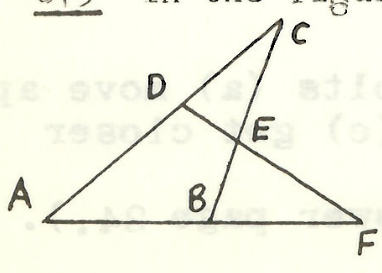
In this G-L square put, for example, $a = 1$, $b = 2$, $c = 4$, $d = 5$, $e = 3$, $\alpha = 10$, $\beta = 0$, $\gamma = 5$, $\delta = 15$, $\varepsilon = 20$.

$a+\alpha$	$b+\beta$	$c+\gamma$	$d+\delta$	$e+\varepsilon$
$b+\varepsilon$	$c+\alpha$	$d+\beta$	$e+\gamma$	$a+\delta$
$c+\delta$	$d+\varepsilon$	$e+\alpha$	$a+\beta$	$b+\gamma$
$d+\gamma$	$e+\delta$	$a+\varepsilon$	$b+\alpha$	$c+\beta$
$e+\beta$	$a+\gamma$	$b+\delta$	$c+\varepsilon$	$d+\alpha$

Correct: P.O'Sullivan; S.Byrnes (Macquarie B.H.S.); P.Jablon; J.Tong; W.Bundschuh.

....

075 In the figure shown, it is known that AB* + BE* = AD* + DE*. Prove that AC* + CE* = AF* + FE*. [Here AB* stands for the length of the interval AB etc.]



Answer: First note that if in Fig. 1 AB* + BE* = AD* + DE* and also D'E' is parallel to DE then AB* + BE'* is not equal to AD'* + D'E'*. (After constructing E'X parallel to DD' this statement is easily seen to be equivalent to the known fact that E'X* is not equal to EX* + EE'*). To say the same thing a different way if it is known that AB* + BE'* = AD'* + D'E* and D'E' ∥ DE then D'E' in fact coincides with DE.

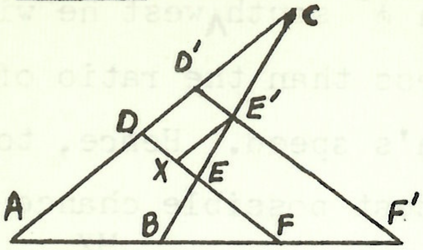


Fig 1.

Now construct the outcircle of Δ ABC shown in Fig. 2 and let D'F' be parallel to DF and tangential to the circle. Let AC, BC, D'F' and AB touch the circle at W, X, Y and Z respectively. Then AB* + BE'* = AB* + BX* - E'X* = AB* + BZ* - E'X* = AZ* - E'X* = AW* - E'Y* = AD'* + D'W* - E'Y* = AD*' + D'Y* - E'Y* = AD'* + D'E'*.

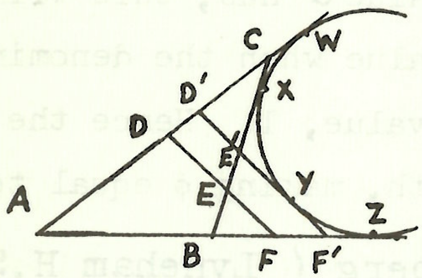


Fig 2.

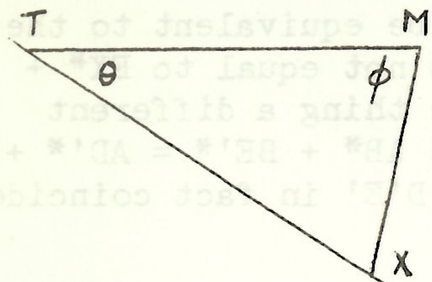
Therefore, by the initial paragraph, D'E' coincides with DE. Consequently AC* + CE* = AC* + CX* + EX* = AF'* + F'Z* + E'X* = AF'* + F'Y* + E'Y* = AF'* + E'F'*
= AF* + EF*.

Correct answers received from: J. Tong (an excellent trigonometric proof)

....

076 A man lost in the desert hears a train whistle due west of him. Although the track is too far away to be seen he knows that it is straight, and runs in a direction somewhere between south and east, (but does not know its exact course). He realises that his only chance to avoid perishing from thirst is to reach the track before the train has passed. In which direction should he travel to give himself the greatest possible chance of survival? (You are expected to assume that the man and the train both move at constant speeds, the former more slowly than the latter).

Answer: If the man travels in a direction ϕ^{o} south of west he will catch the train provided $\frac{MX}{TX}$ is less than the ratio of his speed to the train's speed. Hence, to give himself the greatest possible chance of survival he should choose ϕ to make $\frac{MX}{TX}$ as small as possible. By the sine rule $\frac{MX}{TX} = \frac{\sin\theta}{\sin\phi}$ and, whatever value θ has, this will achieve its minimum value when the denominator assumes its greatest value, 1. Hence the man should travel due south, making ϕ equal to 90^{o}.



Correct answers received from: T.Jagtenberg (Lyneham H.S.); P.O'Sullivan: J.Tong: D.Nash (Parkes H.S.): F.Takacs; A.Oberlander: W.Bundschuh; P.Berkemeier.

....

077 If

$$
\begin{aligned}
K = & (x_1 - x_2)^2 + (x_1 - x_3)^2 + \ldots + (x_1 - x_{2m})^2 \\
& + (x_2 - x_3)^2 + (x_2 - x_4)^2 + \ldots + (x_2 - x_{2m})^2 \\
& + (x_3 - x_4)^2 + \ldots + (x_3 - x_{2m})^2 \\
& + \ldots\ldots\ldots \\
& \ldots\ldots\ldots\ldots\ldots\ldots \\
& + (x_{2m-1} - x_{2m})^2, \\
= & \sum_{\substack{i,j=1 \\ i<j}}^{2m} (x_i - x_j)^2
\end{aligned}
$$

find the maximum possible value of K if it is known that each x_i ($i = 1,2,\ldots 2m$) has the value 0 or 1.

Answer: Let r of the symbols x_i have the value 1.

$$K = (2m-1)\sum_{i=1}^{2m} x_i^2 - 2\sum_{\substack{i,j=1\\ i<j}}^{2m} x_i x_j = 2m\sum_{i=1}^{2m} x_i^2 - (\sum x_i^2 + 2\sum x_i x_j)$$

$$= 2m\sum_{i=1}^{2m} x_i^2 - (\sum_{i=1}^{2m} x_i)^2$$

$$= 2m\,r - r^2$$

$$= m^2 - (m-r)^2$$

Hence K achieves its maximum value of m^2 when r is equal to m (so that the non negative term to be subtracted has the value zero).

Correct answers received from: P.Jablon; P.O'Sullivan; J.Tong.

....

078 $P(x)$ is a polynomial whose value is an integer whenever x is an integer greater than 100. Prove that $P(x)$ is also integer valued when x is any integer less than or equal to 100.

Answer: One proof is by induction on the degree of the polynomial. Assume that the statement is true for all polynomials of degree k and let $P(x)$ be a polynomial of degree $(k+1)$. Then $F(x) = P(x+1) - P(x)$ is a polynomial of degree k. [Note that $a_r(x+1)^r - a_r x^r = r a_r x^{r-1} + \ldots$]. If $P(x)$ is an integer whenever $x > 100$ the same is true of $F(x)$, the difference of two integers. By the induction hypothesis $F(x)$ is integer valued for all values of x. Hence $P(100) = P(101) - F(100)$ is an integer, as is $P(99) = P(100) - F(99)$, and in fact it is clear that $P(n)$ is an integer for every integer n. Since we have now shown that if the statement is true for polynomials of degree k it is also true for polynomials of degree $(k+1)$ to complete the proof it only remains to observe that for polynomials of degree 0 (constants) the statement is obvious.

Correct answers received from: P.O'Sullivan. (Others believed that all coefficients must be integral, but this need not be true. Consider $\frac{1}{2}x(x+1)$.)

....

079 Prove that if the polynomial

$$a_0x^7 + a_1x^6 + a_2x^5 + a_3x^4 + a_4x^3 + a_5x^2 + a_6x + a_7$$

whose coefficients are all integers, has for seven different integral values of x the value $+1$ or -1, then it cannot be factorised as the product of two polynomials with integral coefficients.

Answer: It follows from the factor theorem that if a polynomial $P(x)$ has the same value, a, for k different values of x (say $x_1, x_2, \ldots, x_k$) then the degree of $P(x)$ is at least k. In fact $(x-x_1), (x-x_2), \ldots, (x-x_k)$ are all factors of $P(x)-a$ so that $P(x) = a + Q(x)(x-x_1)\ldots(x-x_k)$.

Now suppose that the given polynomial $P(x)$ factorises, $P(x) = F(x)\,G(x)$, where $F(x)$ and $G(x)$ are both polynomials with integral coefficients. There exist seven integers, $(x_1, x_2, \ldots, x_7)$ for which the value $P(x_i) = \pm 1$. That is for each i $F(x_i)\,G(x_i) = \pm 1$. But $F(x_i)$ and $G(x_i)$ are themselves integers, so each must be ± 1 for each value of i. Consider $F(x_i) = \pm 1$ for each of seven integers x_i. Either the plus sign or the minus sign applies at least four times and it follows from the first paragraph that the degree of $F(x)$ is at least 4. Similarly the degree of $G(x)$ is at least 4. Their product must be a polynomial whose degree is at least 8, which contradicts the fact that the given polynomial $P(x)$ has degree 7.

This contradiction shows that no such factorisation is possible.

Correct answers received from: **P.O'Sullivan; H.Clarke (Forest H.S.); D.Nash.**

....

<u>080</u> You are provided with a pan-balance (but no weights) and n objects. Let W_n be the minimum number of weighings of one object against another which enables you to be certain of arranging the objects in increasing order of weight. [Hence $W_2 = 1$, $W_3 = 3$, and $W_4 = 5$]. Show that $W_5 = 7$.

A general formula for W_n is unknown. However, obtain the following upper bound for W_n : -

if $2^{e-1} < n \leqslant 2^e$ then $W_n \leqslant en + 1 - 2^e$

<u>Answer:</u> To show that seven weighings are sufficient to order five objects is quite tricky. Label the objects a, b, c, d and e. First weigh a against b and second weigh c against d. Relabelling if necessary we may assume that a is heavier than b, and c than d. Thirdly weigh a against c. Again we may assume that a is heavier than c since otherwise we can relabel these four objects so that after three weighings the situation is as shown in Fig. 1.

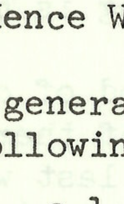


Fig 1

For the fourth weighing weigh e against c. If e is heavier than c, next weight it against a; if it is lighter than c, it should instead be compared with d on the fifth weighing.

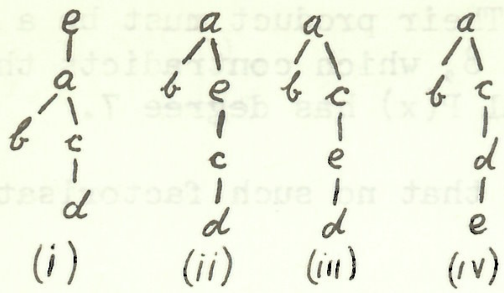


Fig 2

Hence after 5 weighings we arrive at one of the four situations shown in Fig. 2. The last three of these are essentially the same and if one of them applies we can permute the labels of c, d and e until Fig. 2(IV) is obtained.

If Fig. 2(i) applies the sixth weighing maybe b against c, and if b is lighter a seventh weighing of b against d completes the ordering.

If Fig. 2(IV) applies, the sixth weighing must be b against d. If b is heavier the seventh weighing is b against c. If b is lighter than d <u>the seventh</u> weighing is b against e.

To do the second part, note that one method of ordering n weights consists in first ordering (n-1) of them before introducing the last weight at all. This last weight is then weighed against the middle weight of the (n-1) already ordered, (or one of the middle two) then (if it is heavier) against the middle one of the heavier half of the ordered set, and so on. For example if an eighth weight, h, is to be placed amongst seven weights a, b, c, d, e, f, g which are already in decreasing order of weight, first weigh h against d, then against either b or f, after which one final weighing will determine its position precisely. A little investigation shows that e weighings are necessary and sufficient to place the nth weight correctly if $2^{e-1} < n \leq 2^{e}$.

To order the weights one could use this technique systematically, adding one weight at a time and placing it correctly amongst those previously ordered. It is easy to calculate the number of weighings required. For example, if n = 13 the number of weighings is

$$\underset{(2)}{1} + \underset{(3)}{2} + \underset{(4)}{2} + \underset{(5)}{3} + \underset{(6)}{3} + \underset{(7)}{3} + \underset{(8)}{3} + \underset{(9)}{4} + \underset{(10)}{4} + \underset{(11)}{4} + \underset{(12)}{4} + \underset{(13)}{4} = 37.$$

(Here, the entry $+\underset{(11)}{4}$, for example, indicates that it takes an additional 4 weighings to place the eleventh coin correctly after the first ten have been ordered).

In fact for a general n ($a^{e-1} < n < 2^e$) the number of weighings required by this system is

$$1 + 2 \times 2 + 3 \times 4 + 4 \times 8 + \ldots + r \times 2^{r-1} + \ldots + (e-1) \times 2^{e-2} + e \times (n - 2^{e-1}).$$

Since $\sum_{r=1}^{k} rx^{r-1} = \frac{kx^k}{x-1} - \frac{x^k - 1}{(x-1)^2}$ (the derivative of $\sum_{r=0}^{k} x^r = \frac{x^{k+1} - 1}{x - 1}$) putting k = e - 1 and x = 2 gives

for the number of weighings $(e-1)2^{e-1} - (2^{e-1} - 1) + e(n - 2^{e-1})$

$= e\,n + 1 - 2^e$. Hence the result stated.

Correct answers received from: **P.O'Sullivan (a brilliant answer. J.Tong succeeded with the first part, P.Jablon and D.Nash were substantially right with the second.**

....

The Careless Teller. (Answer p.24)

A careless teller interchanged the dollars and cents when he cashed a cheque for Mr. Smith. After buying an 18c. train fare home, Smith, who had not checked his money at the bank, was surprised to find that he had three times the value of the original cheque. What was the amount of the cheque?.

Answers to the Puzzles.

Bolts. (p 15). The heads of the bolts stay the same distance apart.

A Chessboard Problem. (p 12). The largest possible such circle has its centre at the middle of a black square and has a radius of $\sqrt{10}$ inches.

The Careless Teller (p23) Let the amount of the check be x dollars and y cents. Smith received (100 y + x) cents instead of (100 x + y) cents. We require positive integers x and y less than 100 such that $(100y + x) - 18 = 3(100x + y)$.

Rearranging gives $97(y - 3x) = 8x + 18$. Since the R.H.S. is even, $2 \mid (y-3x)$. Since the R.H.S. is less than 818, $(y - 3x)$ is equal to 2, 4, 6, or 8. Only the first of these yields an integral value for x, viz $x = 22$. Then $y = 3x + 2 = 68$, and the amount of the original cheque was $22-68.

The Three Applicants. Answer. (For puzzle, see p 9).

Call the applicants A, B and C, the successful one being A. It is clear that any applicant seeing two white markers would know immediately that his own was black. A argues that if his marker were white, B, seeing one white and one black marker, would quickly have deduced that his own was not also white, since C was clearly unable to decide rapidly the colour of his marker. But B is still puzzled. Therefore A's marker cannot be white.