SCHOOL COMPETITION 1970

SOLUTIONS

JUNIOR DIVISION

1. Solve for x, y and z the simultaneous system of equations

$$x(x+y) + z(x-y) = a,$$
$$y(y+z) + x(y-z) = b,$$
$$z(z+x) + y(z-x) = c,$$

where a, b, and c are given positive integers.

What can you say if both positive and negative values are allowed for a, b and c?

Answer Adding (i) and (ii) after expanding the left hand sides, we get $x^2 + y^2 + 2xy = a + b$ so $x + y = \pm\sqrt{a + b}$. Similarly (ii) and (iii) yield $y + z = \pm\sqrt{b + c}$, (iii) and (i) give $z + x = \pm\sqrt{c + a}$. Consequently our answers are

$$x = \tfrac{1}{2}\{\pm\sqrt{a + b} \mp \sqrt{b + c} \pm \sqrt{c + a}\};$$
$$y = \tfrac{1}{2}\{\pm\sqrt{b + c} \mp \sqrt{c + a} \pm \sqrt{a + b}\}$$
$$z = \tfrac{1}{2}\{\pm\sqrt{c + a} \mp \sqrt{a + b} \pm \sqrt{b + c}\}.$$

Whatever a, b and c are we must have $a+b \geq 0$, $b+c \geq 0$, $c+a \geq 0$ so that no more than one can be negative. For example, if $c < 0$, then we must have $a \geq |c|$, $b \geq |c|$.

2.(i) Which of the following four statements are true, and which are false? If a statement is false, give an example showing this. If the statement is true prove it.

(a) If a polygon inscribed in a circle has equal sides then it has equal angles.

(b) If a polygon inscribed in a circle has

equal angles then it has equal sides.
(c) If a polygon has an inscribed circle (all its sides are tangent to the circle) and its sides are equal then its angles are also equal.
(d) If a polygon has an inscribed circle and its angles are equal then also its sides are equal.

2.(ii) Let n be the number of sides of the polygon. For what values of n are all four statements true?

Answer (i) (a) True. In Fig.1 triangles OAB, OBC are congruent (S,S,S). As all such triangles are isosceles we get: $\frac{1}{2}\angle A = \frac{1}{2}\angle B = \frac{1}{2}\angle C = \ldots$.

(b) False. A non-square rectangle can always be inscribed in a circle.
(c) False. A circle can always be inscribed in a non-square rhombus.
(d) True. See Fig.2. As radii and tangents are perpendicular at points of contact we have right angles at P and Q. This gives triangles OAP, OBP congruent as OA must bisect angle ZAB (tangents from a point theorem) etc. So P is midpoint of AB, and likewise Q is midpoint of BC. Then we get triangles OPB, OQB congruent so $\frac{1}{2}AB = \frac{1}{2}BC$ etc. The congruency theorem used is the right-angled triangle one in both cases.

Answer (ii) True for n odd.
(b) See Fig. 3. Let equal angles = w^o. Each triangle is isosceles so angles are as shown whence we can prove alternate triangles congruent.
(c) As in (b). See Fig.4. Let lengths of equal sides be k units. Tangents from a point are equal so triangles OPB, OQB are congruent and OB bisects angle B etc. This leads to $\angle B = \angle D$, $\angle A = \angle C$; i.e. every other angle equal.

(See diagrams over page)

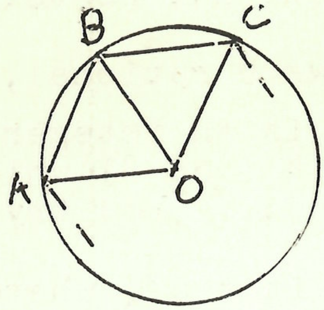


Fig.1

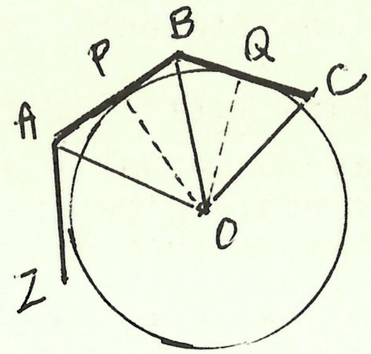


Fig.2

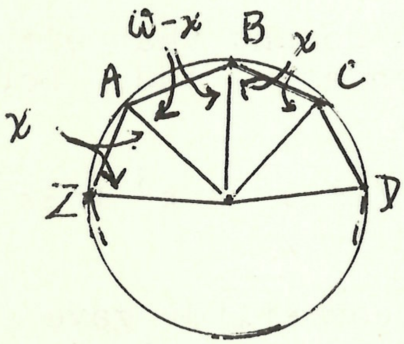


Fig.3

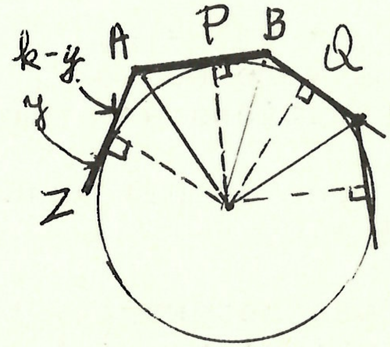


Fig.4

3. Prove that amongst any six consecutive positive integers there is always at least one which has no factor (apart from 1) in common with any of the others.

Answer The 6 integers must all have different remainders when divided by 6. The two that have remainders 1 and 5 must have a difference of either 4 or 2, so that any common divisor of these two integers must divide 2. As, however, the two integers are odd this shows that their g.c.d. = 1. Neither of them can be divisible by 3; if one of them is divisible by 5 the other isn't. So the one not divisible by 5 is not divisible by any integer less than 7 (except 1). As its "distance" from any of the other 5 is less than 7, the result follows.

4. Show that the roots of the equation $ax^2 + bx + c = 0$ must be irrational if a, b and c are all odd integers. Find a cubic equation with odd

integer coefficients which has rational roots.

<u>Answer</u> Part 1.(i) If the roots were rational, we would have $b^2 - 4ac = d^2$ where d is an integer. But this would give $(b-d)(b+d) \equiv 4 \pmod 8$ which is impossible whether d is even or odd. (You try it.)

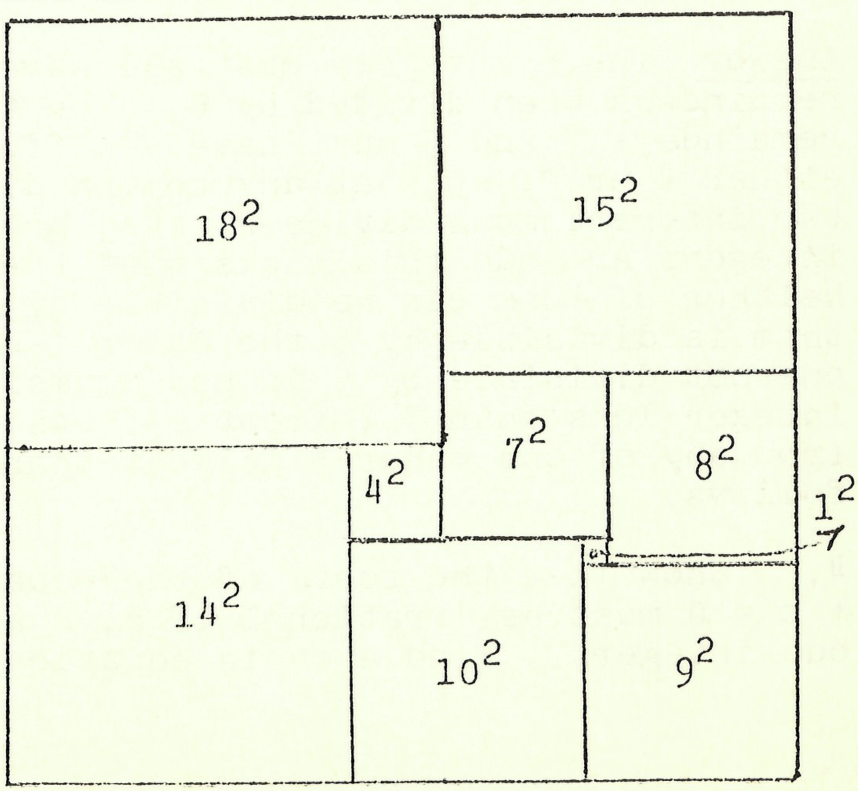
(ii) If a root is rational then it = p/q where p, q are integers, q non-zero. This gives $ap^2 + bpq + cq^2 = 0$. This would demand that either all terms were odd or 2 were even and 1 odd. Both these conditions are impossible.

Part 2. One simple answer is: $(x+1)^3 = x^3 + 3x^2 + 3x + 1 = 0$.

It was noteworthy that one competitor gave both solutions to Part 1 for which he received bonus marks.

5. You are given nine squares with sides 1, 4, 7, 8, 9, 10, 14, 15, 18 units respectively. These can be put together (without overlapping) so as to fill out a rectangle. Find the dimensions of the rectangle and show how it can be formed from the above squares.

<u>Answer</u>

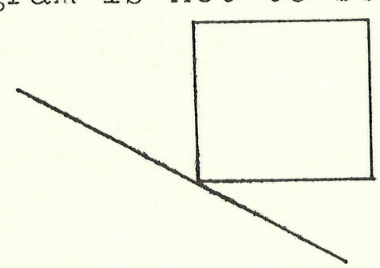


Area of rectangle = $1^2+4^2+7^2+8^2+9^2+10^2+14^2+15^2+18^2$ sq.units = 1,056 sq.units. $1{,}056 = 2^5 \times 3 \times 11 = xy$ where $x \geq 18$, $y \geq 18$. But one of them, say x, must be greater than or equal to $18+15 = 33$. So we have only to consider (i) 33×32 (ii) 44×24 (iii) 48×22. (ii) gives a subrectangle of side 6 to fill in - impossible. (iii) would give a subrectangle 18×4 to fill in - impossible. (i) then gives diagram shown.

Few competitors revealed how they thought out the diagram, unfortunately.

6. A rectangle has sides m units and n units where m and n are positive integers with no common factor. The rectangle is subdivided into mn unit squares and a diagonal of the rectangle drawn. Find a formula for the number of squares traversed by the diagonal.

Can you find a formula which will apply when m and n have a common factor? A square which has only a corner lying on the diagonal as in the accompanying diagram is not to be counted.



Answer First Part. See Fig.1. From similar triangles, we can prove that the only corners which the diagonal meets are the 2 opposite corners of the rectangle. Every time the diagonal cuts through a vertical line it enters a new square and similarly for horizontal lines. It cuts through $(m-1)$ vertical lines and $(n-1)$ horizontal lines; this total, with the square the diagonal starts from, gives the answer as $(m-1) + (n-1) + 1 = m + n - 1$.

Second Part. See Fig.2. Every time the diagonal cuts through a corner, it enters a new $\frac{m}{d}\times\frac{n}{d}$ rectangle where d is the highest common factor of m and n. As the diagonal passes through d such $\frac{m}{d}\times\frac{n}{d}$ rectangles the formula is: No. of corners = $d(\frac{m}{d} + \frac{n}{d} - 1) = m + n - d$.

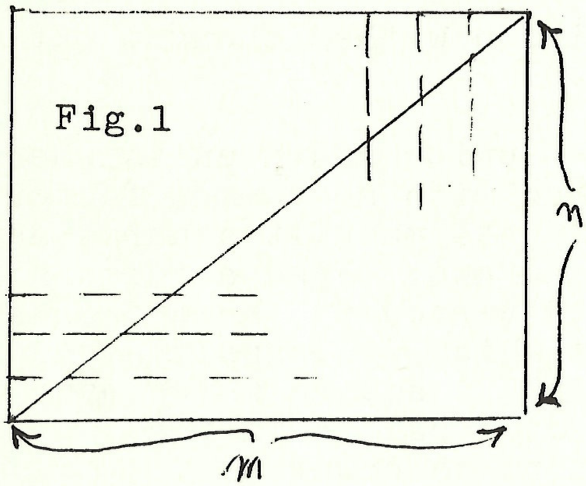


Fig.1

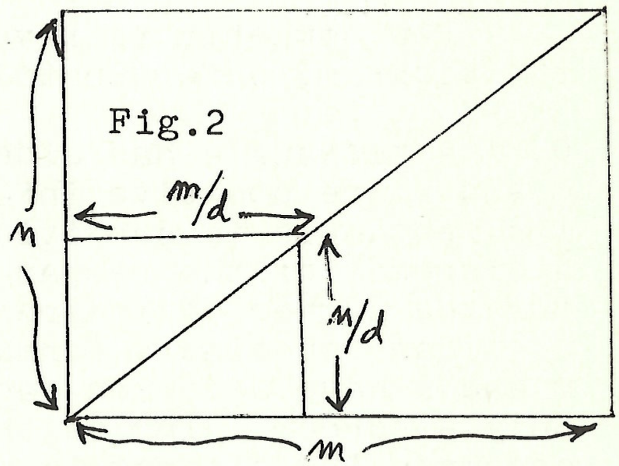


Fig.2

SENIOR DIVISION

1.(i) Prove that amongst any six consecutive positive integers there is always at least one which is relatively prime to each of the others.

(ii) Prove that amongst any eleven consecutive integers there is always one which is relatively prime to each of the others.

Answer (i) See Question 3, Junior Division.

Answer (ii) Let the integers be $a_1, a_2, \ldots, a_{11}$ with $a_{i+1} = a_i+1$. There are 6 sets of 6 consecutive integers starting with $a_1, a_2, \ldots, a_6$ respectively each of which contains an integer b_j $(j = 1,2,\ldots,6)$ which is prime to the other 5 by (i). (We may have $b_1 = b_2$ etc.) Consequently none of the b's is divisible by 2, 3 or 5 so that the only integer less than 11 which may divide the b's is 7. If there is only one b_j it must be a_6 and so prime to all other a_i. If there are two or more b's and one is not divisible by 7 then this is the integer we want. If all the b's are divisible by 7, then there are exactly two b's and one must be even, which is a contradiction.

2.(i) If $P(x)$ is a polynomial of degree n whose value is an integer for $x = 0$, $x = 1$, $x = 2$, ... , $x = n$, then $P(x)$ is an integer when x is any integer. Prove this.

(ii) If (i) is altered by replacing 0, 1, 2, ... ,n by any other n+1 consecutive integers, the statement remains true. Prove this.

(iii) $F(x)$ is a polynomial of degree n whose value is an integer for $x = 0$, $x = 1^2$, $x = 2^2$, ... , $x = n^2$. Prove that $F(x)$ is an integer when x is any perfect square.

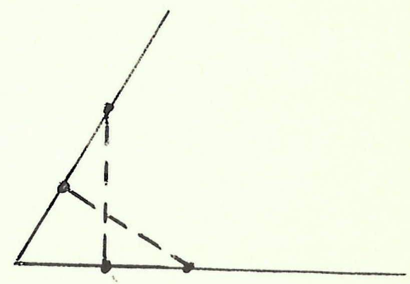
Answer (i), (ii). Both these parts can be proved for all $P(x)$ simultaneously by induction on n as follows:

Both are clearly true when $n = 0$, for then $P(x)$ is a constant. But $Q(x) = P(x+1) - P(x)$ has degree 1 less than that of P, and takes integer values for suitably many integers x. So induction shows that $Q(x)$ is an integer for each integer x. As $P(k)$ is an integer for some k, a further inductive argument shows that $P(k+r) + P(k-r)$ are integers for each positive integer r.

Answer (iii) A similar argument, using F instead of P and (ii) instead of (i) proves this part.

3. Suppose you are provided with a straight edge with two marks on it, one unit apart. The only operations allowable with this instrument are (a) ruling straight lines and (b) marking a point on a given straight line at a unit distance from a given point on the line. (You are not given a compass.) Using this instrument show how to perform the following constructions: (i) Bisect a given angle; (ii) Construct a square whose diagonals meet at a given point; (iii) Construct a square with one vertex at a given point.

Answer For part (i), mark off two points at unit distance along each arm of the angle and construct the bisector as shown in the diagram. This method fails for an angle of 180°. Can anyone construct a perpendicular to a given line at a given point with this instrument?



For part (ii), draw any two lines through the point and then bisect both angles to obtain a pair of perpendicular lines. Mark off one unit along each perpendicular to obtain the corners of the square.

For part (iii), bisect the angles made by the diagonals of the square obtained in part (ii).

4. A "graph" is a collection of a finite number of points (called "vertices") some pairs of which are joined by curves (called "edges"). There is at most one edge joining any two vertices. The only vertices on an edge are its two end points.

A graph contains a circuit of length r if there are r vertices $P_1, P_2, \ldots, P_r$ such that each of the pairs $P_1P_2, P_2P_3, \ldots, P_{r-1}P_r; P_rP_1$ is joined by an edge. Prove that if in a graph each vertex is an end point of at least k edges ($k \geq 2$) then the graph contains a circuit of length at least $k+1$.

<u>Answer</u> One candidate gave this quite simple argument. It is possible to obtain a sequence Q_1, Q_2, ... , Q_{k+1} of distinct points such that the pairs Q_1Q_2, Q_2Q_3, ... , Q_kQ_{k+1} are each joined. If Q_{k+1} is joined to Q_1, the required circuit has been found. If not, there is a Q_{k+2}, not identical with any Q_i, $i < k+2$, such that Q_{k+1} is joined to Q_{k+2}. If Q_{k+2} is joined to Q_1 or Q_2, the circuit has been found. If not, there is a Q_{k+3}, not identical with any Q_i, $i < k+3$, such that Q_{k+2} is joined to Q_{k+3}. And this argument may be continued only until the number of points have all been used. By that time a circuit will have been found.

5. Given a finite set S of positive integers, we call the <u>order</u> of any element x of S the length of the longest sequence of distinct elements of S starting with x, e.g. x, y, ... , z such that each member of the sequence is a multiple of the preceding one. For example if

$$S = \{2,4,5,7,12,15,16\}$$

then the order of 2 is 3 since 2, 4, 12 or 2, 4, 16 have 3 members and there are no longer sequences of this kind starting with 2. Similarly the order of 5 is 2, the order of 12 is 1.

Assuming that S has 101 elements and no element of S has an order exceeding 10, prove that there is a subset T of S containing at least 11 elements such that no element of T is divisible by any other element of T.

We welcome any generalizations of this result.

Answer As there are only 10 possible orders but 101 elements there must be 11 elements of the same order, say of order r. No one of these 11 elements can divide another similar one because if element a divides element b then the order of b is less than that of a.

MAGIC SQUARES

27	29	2	4	13	36
9	11	20	22	31	18
32	25	7	3	21	23
14	16	34	30	12	5
28	6	15	17	26	19
1	24	33	35	8	10

A Chinese magic square of about A.D.1590.

A magic square has numbers such that the sum of each row, each column and each diagonal is the same.

Can you fill in this magic square using the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, each of them only once, so that the sum of each row, each column and both diagonals is 15?
