ANSWERS TO PROBLEMS IN PARABOLA VOL.6, NO.1

J131 (i) How many lines equidistant from three given points can be drawn in the plane?

(ii) Four non-coplanar points are given. How many planes equidistant from all four points can be found?

Answer (i) It is intuitively obvious (and very easy to supply a geometrical proof) that a line ℓ is equidistant from two given points A and B if and only if either ℓ is parallel to AB, or ℓ passes through the mid-point of AB. [In fig. 1, prove that ABXY is

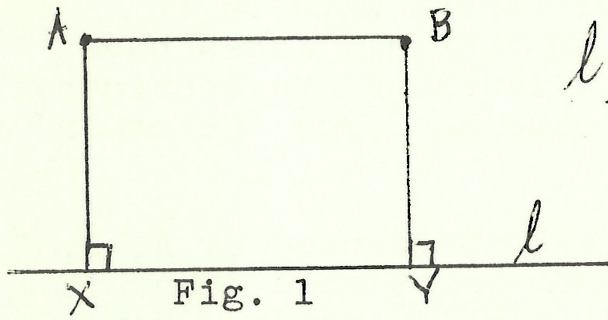


Fig. 1

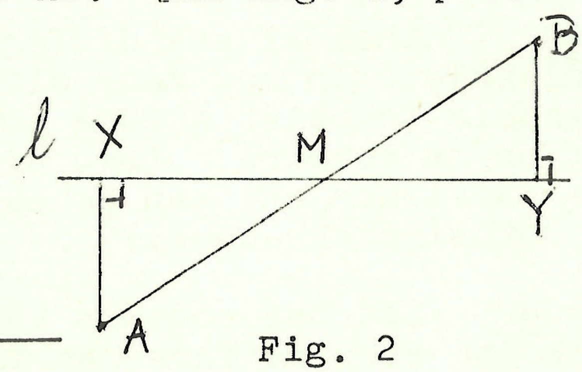


Fig. 2

a rectangle, and in fig. 2 prove that ΔAMX is congruent to ΔBMY]. With this preparation, it is easy to answer the given question. If the three given points are collinear any line parallel to the line on which they lie is equidistant from all three. Otherwise, the given points form a triangle and there are only three allowable lines, namely, the joins of the midpoints of the sides of the triangle.

Answer (ii) A plane p is equidistant from two points A and B if and only if either it is parallel to the line AB or it passes through the midpoint of AB. Using this one sees easily that p is equidistant from three points A, B and C if and only if either p is parallel to the plane ABC or p contains two of the points L, M and N, the midpoints of BC, CA and AB respectively. With this preparation the given question can be answered as follows. If two of the points A and B lie on one side of p and the other two C and D lie on the other, then p is the plane determined by the midpoints X, Y and Z of AC, AD and DB respectively. Since there are three ways of grouping the four points into two pairs (AB,CD; AC,BD; and AD,BC), three planes are obtainable in this way. If three points (A, B and C, say) lie on one side of p and one (D) on the other then p is the plane (parallel to plane ABC) determined by the midpoints of DA, DB and DC. There are four planes obtainable in this way, one for each of the groupings ABC,D; ABD,C; ACD,B; and BCD,A. Thus in all seven planes can be found.

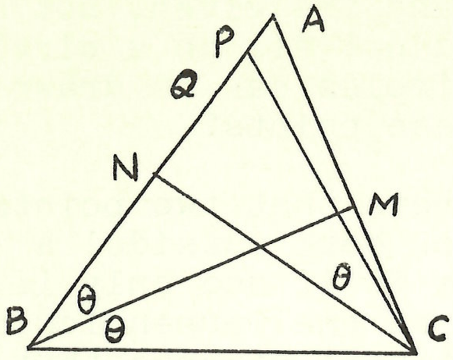
J132 Paints of six different colours are available. In how many different ways is it possible to paint a cube so that all faces have different colours. (Two colourings are considered the same when one can be obtained from the other by rotating the cube.)

Answer Let the colours be represented by A, B, C, D, E and F. One face has to be painted A and it makes no difference which face is chosen. Now there are five remaining colours which can be used on the opposite face. Once one has been chosen (F, say) any one of the remaining faces can be painted with a third colour (B) and there are then 3.2.1 = 6 different ways of colouring the remaining surfaces (with C, D and E).

Hence altogether there are 5×6, viz 30, different colourings of the cube.

O133-4 Solve the problems at the end of the article concerning isosceles triangles.

Answer O133 In following through the suggestions in the hint one needs to be aware that $\angle ACN$ =

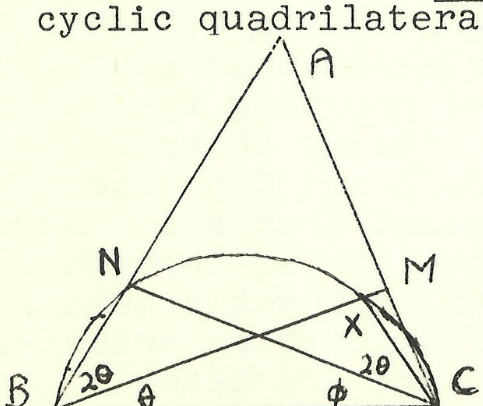


$\frac{1}{2}\angle C > \frac{1}{2}\angle B$ to ensure the existence of P on AN, and then that PB is longer than PC in PBC (being opposite a larger angle in the triangle) to ensure the existence of Q on PB (whether it lies on PN or NB is irrelevant. Hence M and Q must lie on opposite sides of the line CP and it is impossible that QM could be parallel to PC. However, in ΔQBM and ΔPCM we have

$$*QB = *PC \text{ (construction)}$$
$$\angle B = \angle C \text{ (construction)}$$
$$*\overline{BM} = *\overline{CN} \text{ (given)}$$

whence the triangles are congruent, and $\angle BQM = \angle NPC$. This makes QM parallel to PC, as corresponding angles have been proved equal. Our assumption that the base angles $\angle ABC$ and $\angle ACB$ are unequal has now led us to a clear contradiction, so we conclude that they must be equal and the triangle isosceles.

Answer 0134 Construct CX so that $\angle XCN = \frac{2}{3}\angle B$ intersecting BM at X. Since $\angle MCN = \frac{2}{3}\angle C > \frac{2}{3}\angle B$ we must have X on the interval BM (not produced). By our construction $\angle XCN = \angle XBN$ whence BCXN is a cyclic quadrilateral. The chord BX of the circumcircle subtends an angle $\angle XCB = \frac{2}{3}\angle B + \frac{1}{3}\angle C$ at the circumference whilst the chord CN subtends the angle $\angle NBC = \frac{2}{3}\angle B + \frac{1}{3}\angle B$, a smaller angle. It follows that BX is a longer chord than CN and, a fortiori, BM must be longer than CN.



0135 Four points in the plane are given, not all on the same straight line, and not all on a circle. How many straight lines and circles can be drawn which are equidistant from these points?

Answer 0135 It is easy to prove that two points A and B which are both inside (or both outside) a circle $\mathcal{C}$, are equidistant from it if and only if the centre of the circle lies on the perpendicular bisector of AB. Using this we next observe that three points A, B and C, all outside (or inside) a circle $\mathcal{C}$, are equidistant from it if and only if the circle through the three points is concentric with $\mathcal{C}$. (By regarding a straight line as a circle whose centre is at an infinite distance in a direction perpendicular to the line we shall take the above statements to include the case in which $\mathcal{C}$ is a straight line. In the same way we shall not make further reference to straight lines, leaving it to our readers to decide for what given sets of points some of the circles referred to are really lines.) Noting also that four points all inside (or all outside) can be equidistant from a circle only if they are concyclic we can now answer the question asked as follows. If A and B are two of the given points, and C and D the remaining two there is exactly one circle with A and B on one side (inside or outside) and C and D on the other, equidistant from all four points. (Its centre is the intersection of the perpendicular bisectors of AB and CD, its radius is the arithmetic mean of the distances from this point to A and to C.) There are three circles which arise in this way one for each of the three ways of grouping the given four points in two pairs. Again, there is one and only one circle having A, B and C on one side and D on the other, equidistant from all four. (Its centre is the centre O of the circle through A, B and C, and its radius is the arithmetic mean of the lengths of DA and OD.) Four circles arise in this way. Hence altogether there are seven circles having the required property.

0136 In how many ways can postage of 4m cents be paid using 1 cent, 2 cent and/or 4 cent stamps? What about postage of 4m+1 cents, 4m+2 cents and 4m+3 cents?

Answer Postage of 2n cents or of (2n+1) cents can be paid in (n+1) ways using only 1 cent and 2 cent stamps (since the number of 2 cent stamps is any one of 0, 1, 2, ... , n).

If in paying postage of $(4m+\tau)$ cents one uses exactly r four cent stamps, the remaining $4(m-r)+\tau$ cents can be paid in $2(m-r)+1$ ways if τ is 0 or 1, and in $2(m-r)+2$ ways if $\tau = 2$ or 3. We now have to sum for all possible values of r, which can range from 0 to m. The total number of ways is therefore

$$N = \sum_{r=0}^{m} [2(m-r) + 1] \text{ or } \sum_{r=0}^{m} [2(m-r) + 2];$$

namely $N = (m+1)^2$ if τ is 0 or 1
and $N = (m+1)(m+2)$ if τ is 2 or 3.

0137 There are n people at a party, and n handshakes are made (no pair of people shaking hands twice). Prove that, for some r, $3 \leq r \leq n$, it is possible to find r people $P_1, P_2, \ldots, P_r$ such that P_1 has shaken hands with P_2, P_2 with P_3, ... , P_{r-1} with P_r and P_r with P_1.

Answer We shall prove the result by mathematical induction on n. We first observe that if n = 3, the result is true, since as three handshakes have been made each person has shaken hands with both of the others.

Now suppose the statement is true when n = k, and consider a gathering of (k+1) people at which (k+1) handshakes have been made. Let P_1 be a person at the party who has shaken hands; let P_2 be any person who has shaken hands with P_1. If P_2 has not shaken hands with anyone else we proceed as indicated below for P_s. Otherwise let P_3 be any other person who has shaken hands with P_2, and similarly let P_4 be any fourth person who has shaken hands with P_3.

We eventually arrive at a chain of distinct people $P_1, P_2, \dots, P_s$, which can not be further extended, in which each person has shaken hands with his successor in the chain. It cannot be extended either because P_s has not shaken hands with anyone except P_{s-1}, or because every other person (for example, P_ℓ) with whom he has shaken hands is already in the chain. In the second case, the group $P_\ell, P_{+1}, \dots, P_{s-1}, P_s$ is the desired type of cycle. In the first case we simply omit from consideration P_5 and the one handshake he has made. The remaining k people have made k handshakes amongst themselves and by our supposition (the induction hypothesis) the desired result is true. Hence we have shown that the result is true for a group of (k+1) people if it is true for any group of k people, as well as being true for a group of 3 people. It follows that the result holds for any $n \geq 3$.

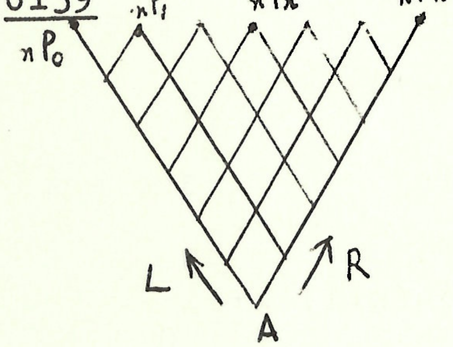
<u>0138</u> There are 2n people at a party, and n^2+1 handshakes are made (no pair of people shaking hands twice). Prove that it is possible to find 3 people, each of whom has shaken hands with the other two.

<u>Answer</u> This is another problem which lends itself to a proof by induction. First we observe easily that when n = 2 (n = 1 is impossible) the statement is true. [If the maximum possible of 6 handshakes are made by the 4 people, all 4 possible "triangles" are complete; and any handshake (between A and B, say) is a "side" of 2 "triangles", ABC and ABD.

Hence if any handshake is omitted, there still remain the other 2 "triangles".] Now assume that the statement is true for $n = k$, and consider a group of $2(k+1)$ people who have made $(k+1)^2 + 1$ handshakes without repetitions. Let A and B be any pair of people apart from B who have shaken hands with A and T be the set of people apart from A who have shaken hands with B. Then either there is some person C who belongs to both sets S and T, or there isn't. In the first case we already have found our three people, A, B and C who have shaken hands with each other. In the second case, we observe that the number of people in S plus the number in T cannot exceed $2k$, so that the number of handshakes made by A and B combined is at most $2k+1$ (the extra 1 being the AB handshake of course). If we omit A and B and all the handshakes they have made the remaining $2k$ people have made at least $[(k+1)^2 + 1 - (2k+1)] = k^2 + 1$ handshakes amongst themselves, so by our assumption (the induction hypothesis) there must be three people amongst them each of whom has shaken hands with both the others.

We have now shown that if the statement is true for $n = k$ it is true for $n = k+1$, and that it is true when $n = 2$. Hence it is true for all $n \geq 2$.

0139



A network of roads is shown in the diagram. A group of 2^n people leave A, half going in the direction L and half in the direction R. At the first intersection each group splits up, half going in direction L and half in direction R. The same behaviour is repeated at each intersection. Let $P_0, P_1, \ldots, P_n$ be the nth row of intersections. How many people reach each point P_i?

Answer We shall slightly change the notation, calling the nth row of intersections nP_0, nP_1, ... , nP_n, and for any r, $1 \leq r \leq n$, the r+1 points which are the rth row of intersections will be labelled rP_0, rP_1, ... , rP_r. We now claim that the number of people reaching the intersection rP_i is $2^{n-r}\ {}^rC_i$ where ${}^rC_i = \dfrac{r!}{(r-i)!\ i!}$ *, (the answer asked for then being simply ${}^nC_i$). Our claim is clearly correct when r = 1, since ${}^1C_0 = {}^1C_1 = 1$. Let us assume that the formula given for rP_i is correct when $r = k\ (< n)$, that is, for the kth row of intersections; and consider any point $k{+}1\ P_i$ in the (k+1)th row of intersections. Except for the two end points (i = 0 and i = k+1) the people reaching $k{+}1\ P_i$ are half of those who reached kP_i and half of those who reached kP_{i-1}. Hence the number reaching $k{+}1\ P_i = \tfrac{1}{2}\ 2^{n-k}({}^kC_i) + \tfrac{1}{2}\ 2^{n-k}\ {}^kC_{i-1}$

$$= 2^{n-(k+1)}\left[\frac{k!}{(k-i)!\ i!} + \frac{k!}{(k-i+1)!\ (i-1)!}\right]$$

$$= 2^{n-(k+1)}\ \frac{k!}{(k-i+1)!\ i!}\ (\overline{k-i+1} + i)$$

$$= 2^{n-(k+1)}\ \frac{(k+1)!}{(k+1-i)!\ i!} = 2^{n-(k+1)}\ {}^{k+1}C_i .$$

If i = 0, the people reaching $k{+}1\ P_0$ are half of those who reached kP_0, namely $\tfrac{1}{2}\ 2^{n-k}\ {}^kC_0$

$$= 2^{n-(k+1)}\ {}^{k+1}C_0$$

* The factorial function ( )! is defined for non negative integers by 0! = 1 and, if ℓ is a positive integer, by

$$\ell! = 1 \times 2 \times \ldots \times \ell .$$

since ${}^{k}C_0$ and ${}^{k+1}C_0$ are both equal to 1. A similar calculation shows that the number reaching k+1 P_{k+1} is $2^{n-(k+1)}\ {}^{k+1}C_{k+1}$.

Hence if the stated formula for the number of people reaching rP_i is correct whenever $r = k$ ($< n$), we have shown that it is also correct when $r = k+1$. Since it is correct for $r = 1$ it is true for all values of r up to n.

<u>0140</u> Suppose A and B are two equally strong tennis players. Is it more probable that A will beat B in there sets out of 4 or in 5 sets out of eight?

<u>Answer</u> If A and B play n sets, the probability that A will win some specified r sets (for example, the first r sets) and lose the remaining n-r is $(\frac{1}{2})^n$. (His probability of winning one set being $\frac{1}{2}$, his probability of winning the first r sets $(\frac{1}{2})^r$, his probability of losing the last (n-r) sets is $(1-\frac{1}{2})^{n-r}$ $= (\frac{1}{2})^{n-r}$, and multiplying these gives the stated result.) Hence the probability of A's beating B in exactly r sets out of n is ${}^{n}C_r\ (\frac{1}{2})^n$ (where ${}^{n}C_r$ is the number of different ways one can choose r objects from n given objects - in this case sets of tennis). Hence A's probability of winningexactly 3 sets out of 4 is ${}^{4}C_3\ (\frac{1}{2})^4 = 4.\frac{1}{2^4} = \frac{1}{4}$ whilst his probability of winning exactly 5 sets out of 8 is

$${}^{8}C_5\ (\tfrac{1}{2})^8 = \frac{8.7.\cancel{6}.\cancel{5}.\cancel{4}}{\cancel{5}.\cancel{4}.\cancel{3}.\cancel{2}.1} \cdot \frac{1}{2^8}$$

$$= \frac{7}{32},$$ a somewhat smaller expectation. Of course, A's probability of winning "at least" 5 sets out of 8 is considerably larger than his probability of winning at least 3 out of 4 (the probabilities are $\frac{93}{256}$ and $\frac{5}{16}$ ($= \frac{80}{256}$) respectively).

ANSWERS TO SHORT PROBLEMS

The Four 4's (from page 6)

$4 = \frac{4+4}{4} + \sqrt{4}$, $5 = \sqrt{4} + \sqrt{4} + \frac{4}{4}$, $6 = \frac{4+4}{4} + 4$,

$7 = 4 + 4 - \frac{4}{4}$, $8 = \sqrt{4} + \sqrt{4} + \sqrt{4} + \sqrt{4}$.

The Quartz Track (from page 13)

Answer: He drinks $2\pi n \times s \times t = 2\pi nts$ (2 pints)
$= 1$ quart.