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On a recursive fraction operation which leads to irrational numbers and Fibonacci numbers

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Introduction

In the 3rd grade, I was taught fractions in the usual way. I thought: Why not try other ways to calculate fractions instead? In this paper, I describe a recursive fraction operation which in interesting ways leads to irrational numbers and Fibonacci numbers.

The operation

Consider the fraction $\frac{x}{y}$. If we add this same fraction to the numerator x and to the denominator y, we get

$$\frac{x+\frac{x}{y}}{y+\frac{x}{y}}\,.$$

If we keep adding $\frac{x}{y}$ to the inner-most numerators and denominators of the resulting fractions, then we get a sequence of fractions

$$\frac{x}{y} \to \frac{x+\frac{x}{y}}{y+\frac{x}{y}} \to \frac{x+\frac{x+\frac{x}{y}}{y+\frac{x}{y}}}{y+\frac{x+\frac{x}{y}}{y+\frac{x}{y}}} \to \frac{x+\frac{x+\frac{x+\frac{x}{y}}{y+\frac{x}{y}}}{y+\frac{x+\frac{x}{y}}{y+\frac{x}{y}}}}{y+\frac{x+\frac{x+\frac{x}{y}}{y+\frac{x}{y}}}{y+\frac{x+\frac{x}{y}}{y+\frac{x}{y}}}} \to \cdots$$

After n of these operations, we get a fraction $r_n = \frac{N_n}{D_n}$ where N_n and D_n denote the numerator and denominator of r_n, respectively, and where $r_0 = \frac{x}{y}$, $N_0 = x$, and $D_0 = y$, and where

$$r_{n+1} = \frac{x+r_n}{y+r_n} = \frac{N_{n+1}}{D_{n+1}}\,, \tag{1}$$

and

$$N_{n+1} = x + \frac{N_n}{D_n} \qquad \text{and} \qquad D_{n+1} = y + \frac{N_n}{D_n}\,. \tag{2}$$

If the sequence converges, then we get a limit r_∞.

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From fractions to irrational numbers

By choosing different values of x and y, we find interesting sequences. For instance, set $x = 1$ and $y = 3$. Then

$$r_0 = \frac{1}{3}\,, \qquad r_1 = \frac{1+\frac{1}{3}}{3+\frac{1}{3}} = \frac{2}{5}\,, \qquad \text{and} \qquad r_2 = \frac{1+\frac{1+\frac{1}{3}}{3+\frac{1}{3}}}{3+\frac{1+\frac{1}{3}}{3+\frac{1}{3}}} = \frac{1+\frac{2}{5}}{3+\frac{2}{5}} = \frac{7}{17}\,.$$

By calculating more terms of this sequence[2], we see that

$$\frac{1}{3} \to \frac{2}{5} \to \frac{7}{17} \to \frac{12}{29} \to \frac{41}{99} \to \frac{70}{169} \to \frac{239}{577} \to \frac{408}{985} \to \frac{1393}{3363} \to \cdots\,.$$

This sequence converges to

$$r_\infty = \frac{1+\frac{1+\frac{1+\frac{1+\cdots}{3+\cdots}}{3+\frac{1+\cdots}{3+\cdots}}}{3+\frac{1+\frac{1+\cdots}{3+\cdots}}{3+\frac{1+\cdots}{3+\cdots}}}}{3+\frac{1+\frac{1+\frac{1+\cdots}{3+\cdots}}{3+\frac{1+\cdots}{3+\cdots}}}{3+\frac{1+\frac{1+\cdots}{3+\cdots}}{3+\frac{1+\cdots}{3+\cdots}}}} = 0.41421356237309503\ldots = \sqrt{2}-1\,.$$

Here, we started with a rational number $\frac{1}{3}$ but ended with an irrational number $\sqrt{2}-1$.

Let us try another two numbers, $x = 1$ and $y = 5$. We get the sequence

$$\frac{1}{5} \to \frac{3}{13} \to \frac{4}{17} \to \frac{21}{89} \to \frac{55}{233} \to \frac{72}{305} \to \frac{377}{1597} \to \frac{987}{4181} \to \frac{1292}{5473} \to \cdots\,.$$

This sequence converges to

$$r_\infty = \frac{1+\frac{1+\frac{1+\frac{1+\cdots}{5+\cdots}}{5+\frac{1+\cdots}{5+\cdots}}}{5+\frac{1+\frac{1+\cdots}{5+\cdots}}{5+\frac{1+\cdots}{5+\cdots}}}}{5+\frac{1+\frac{1+\frac{1+\cdots}{5+\cdots}}{5+\frac{1+\cdots}{5+\cdots}}}{5+\frac{1+\frac{1+\cdots}{5+\cdots}}{5+\frac{1+\cdots}{5+\cdots}}}} = 0.23606797749978967\ldots = \sqrt{5}-2\,.$$

Again, we started with a rational number but ended with an irrational number.

For the numbers $x = 2$ and $y = 1$, we get a very simple irrational number:

$$2 \to \frac{4}{3} \to \frac{10}{7} \to \frac{24}{17} \to \frac{58}{41} \to \frac{140}{99} \to \frac{338}{239} \to \frac{816}{577} \to \frac{1970}{1393} \to \cdots \to \sqrt{2}\,.$$

[2]For the calculations in this paper, I programmed using Java and the Eclipse Jee Neon IDE. If you want to see the source code of my program, then please go to the URL: https://github.com/AshwinSivakumar/Project-nest

A surprising connection to famous numbers

Now, let's try the simple values $x = 1$ and $y = 2$. We get the sequence

$$\frac{1}{2} \to \frac{3}{5} \to \frac{8}{13} \to \frac{21}{34} \to \frac{55}{89} \to \cdots .$$

We recognise here the famous Fibonacci numbers F_n

$$\frac{F_2}{F_3} \to \frac{F_4}{F_5} \to \frac{F_6}{F_7} \to \frac{F_8}{F_9} \to \frac{F_{10}}{F_{11}} \to \ \cdots .$$

As Johannes Kepler discovered four centuries ago, these ratios converges to

$$\cfrac{1+\cfrac{1+\cfrac{1+\frac{1+\cdots}{2+\cdots}}{2+\frac{1+\cdots}{2+\cdots}}}{2+\cfrac{1+\frac{1+\cdots}{2+\cdots}}{2+\frac{1+\cdots}{2+\cdots}}}}{2+\cfrac{1+\cfrac{1+\frac{1+\cdots}{2+\cdots}}{2+\frac{1+\cdots}{2+\cdots}}}{2+\cfrac{1+\frac{1+\cdots}{2+\cdots}}{2+\frac{1+\cdots}{2+\cdots}}}} = 0.6180339887498949\ldots = \frac{1}{\varphi},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the famous *golden ratio*.[3]

It it surprising to see Fibonacci numbers appear here but it is also useful: for this sequence, we can determine the exact value of each term r_n, using *Binet's Formula*:

$$F_n = \frac{1}{\sqrt{5}}\left(\varphi^n - \varphi^{-n}\right).$$

In particular,

$$r_n = \frac{F_{2n}}{F_{2n+1}} = \frac{\varphi^{2n} - \varphi^{-2n}}{\varphi^{2n+1} - \varphi^{-(2n+1)}}.$$

Further research

Can you find other values of x and y that lead to interesting sequences and limits? In this paper, I used computational methods to find approximations for the limits. Is there an analytical way to find these? And can you find simple recursive relations to determine the numbers N_{n+1} and D_{n+1} from the numbers N_n and D_n?

[3]See https://en.wikipedia.org/wiki/Golden_ratio.