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Problems 1611–1620

Parabola would like to thank Sin Keong Tong for contributing Problem 1616, and Farid Haggar for Problem 1619.

Q1611 Suppose that the expression

$$\left(1 + x^2 + \frac{1}{x}\right)^{10}$$

is expanded and like terms collected. Find the coefficient of x^3.

Q1612 Find all positive integers x, y such that

$$\frac{x+y}{x^2 - xy + y^2} = \frac{8}{73}\,.$$

Q1613 Prove that any given string of decimal digits occurs (consecutively and in the given order) among the digits of n^2 for some integer n.

Q1614 Let $x_1, x_2, \ldots, x_n$ be n different positive integers in increasing order, and suppose also that $x_n < 2x_1$. That is,

$$x_1 < x_2 < \cdots < x_n < 2x_1\,.$$

Prove that if p is a prime number, s is a non–negative integer and the product $x_1x_2\cdots x_n$ is a multiple of p^s, then the quotient is at least $n!$; that is,

$$x_1x_2\cdots x_n \geq p^s n!\,.$$

Q1615

(a) Show that it is possible to choose a point O inside a square and to draw three rays from O, all separated by equal angles, in such a way that the square is divided into three regions of equal area.

(b) Show that in (a), the point O cannot be the centre of the square.

Q1616 Fourteen circular counters of identical size are available; 9 of them are red and 5 are blue. In how many ways can they be arranged into a stack of 14 counters, if there cannot be more than 3 adjacent counters of the same colour?

Q1617 A **dyadic fraction** is a fraction in which the denominator is a power of 2, that is, a fraction

$$\frac{s}{2^n}$$

where s, n are integers and $n \geq 0$.

(a) Show that the sum, difference and product of two dyadic fractions is always a dyadic fraction.

(b) Find two dyadic fractions whose quotient is not a dyadic fraction.

Now let F be the set of all dyadic fractions,

$$F = \left\{ \frac{s}{2^n} \,\middle|\, s, n \text{ are integers with } n \geq 0 \right\}.$$

If a and b are specific numbers, then we write $aF + b$ for the set of all numbers that can be expressed as $ax + b$, where x is in F. That is,

$$aF + b = \{ ax + b \mid x \text{ is in } F \}.$$

(c) Let a and b be fractions in F. Prove that $aF + b = F$ if and only if a is a power of 2, that is, $a = 2^k$ for some integer k. (Note that k may be positive, negative or zero.)

Comment. In future issues we shall be presenting a series of problems about dyadic fractions, leading to questions which have been found important in a current mathematical research project. We hope that readers will be interested to see that even advanced contemporary mathematics sometimes relies on arguments which are accessible to school students.

Q1618 The points A, B, C are collinear, in that order. There is a circle on diameter AB and a circle on diameter AC. A chord of the larger circle is parallel to AC and tangent to the smaller circle; its length is $2x$. Find the area of the region lying between the two circles.

Q1619 A sequence of numbers $a_0, a_1, a_2, a_3, \ldots$ satisfies the equation

$$a_n = a_{n-1} + a_{n-2} + \lambda a_{n-1} a_{n-2} \qquad \text{for } n = 0, 1, 2, \ldots,$$

where λ is a fixed non–zero real number. Find a formula for a_n in terms of the first two values a_0, a_1.

Q1620 There are m boxes, each containing some beads. A positive integer n is specified, with $n < m$. You are allowed to choose any n of the m boxes and then add one bead to each of the chosen boxes.

(a) Prove that if n and m have no common factor, then it is possible to perform this operation, more than once if necessary, in such a way that all boxes end up with the same number of beads.

(b) If n and m do have a common factor (greater than 1), find an initial distribution of beads such that it is impossible for the above operation, no matter how many times repeated, to result in all boxes containing the same number of beads.