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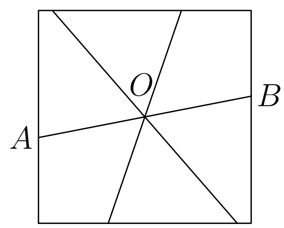
Solutions 1621–1630

Q1621 Take a point O inside a square; from this point draw six rays, all spaced at equal angles. This will divide the square into six regions. Is it possible that all these regions have equal area?

SOLUTION Let the area of the square be 1, and suppose that such an arrangement exists, as shown in the diagram.



There are three regions of area $\frac{1}{6}$ on one side of the line AB, total area $\frac{1}{2}$, and the same on the other; so AB must pass through the centre of the square. The same holds for the other two lines in the "star", and so O must lie at the centre of the square. But then by combining pairs of adjacent regions we obtain an arrangement of three equally spaced rays meeting at the centre of the square and dividing the square into regions of area $\frac{2}{6} = \frac{1}{3}$. We know from the solution to Problem 1615 (see the previous issue of *Parabola*) that this is impossible. Therefore, six rays cannot be arranged as desired.

NOW TRY Problem 1632.

Q1622 Find the sum of the digits of

$$S = 1 + 11 + 111 + 1111 + \cdots + \overbrace{11\cdots 11}^{999\text{ digits}},$$

where the last term on the right hand side has 999 digits, all equal to 1.

SOLUTION Multiply both sides of the equation by 9 to get

$$9S = 9 + 99 + 999 + 9999 + \cdots + \overbrace{99\cdots 99}^{999\text{ digits}}.$$

Now add 1 for each term on the right hand side; since there are 999 terms, this gives

$$\begin{aligned} 9S + 999 &= 10 + 100 + 1000 + 10000 + \cdots + 1\,\overbrace{00\cdots 00}^{999\text{ zeros}} \\ &= \overbrace{11\cdots 11}^{999\text{ ones}}\,0\,; \end{aligned}$$

which can be written with the 1s split into blocks of nine digits,

$$9S + 999 = \overbrace{(111111111)\cdots(111111111)}^{111 \text{ blocks}}0\,.$$

Now 111111111 is exactly divisible by 9, so when we divide both sides by 9 there will be no "carries" between blocks and we get

$$S + 111 = \overbrace{(012345679)\cdots(012345679)}^{111 \text{ blocks}}0\,.$$

The sum of the digits on the right hand side is

$$111 \times (0+1+2+3+4+5+6+7+9) = 4107\,.$$

Subtracting 111 from the right hand side changes the last three digits 790 to 679, and therefore increases the sum of the digits by 6. Hence, the sum of the digits of S is 4113.

Q1623 Consider the product of n factors

$$P_n = 4^{1/4}\,16^{1/16}\,64^{1/64}\cdots(4^n)^{1/4^n}\,.$$

Show that if the number of factors n becomes larger and larger, the product gets closer and closer to a fixed value; and find this value.

SOLUTION We can write the product as

$$P_n = 4^{1/4}\,4^{2/16}\,4^{3/64}\cdots4^{n/4^n}\;;$$

taking logarithms of both sides and then multiplying by 4 yields

$$\begin{aligned}\frac{\log P_n}{\log 4} &= \frac{1}{4} + \frac{2}{16} + \frac{3}{64} + \cdots + \frac{n}{4^n}\\ 4\,\frac{\log P_n}{\log 4} &= 1 + \frac{2}{4} + \frac{3}{16} + \cdots + \frac{n}{4^{n-1}}\,.\end{aligned}$$

By writing $\frac{2}{4} = \frac{1}{4} + \frac{1}{4}$ and $\frac{3}{16} = \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$ and so on, we can re–express the right hand side as

$$\begin{aligned}1 + \frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^{n-1}}&\\ + \frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^{n-1}}&\\ + \frac{1}{16} + \cdots + \frac{1}{4^{n-1}}&\\ + \cdots&\\ + \frac{1}{4^{n-1}}&\end{aligned}$$

Now if n becomes larger and larger, each line of this expression becomes an infinite geometric progression, the first being

$$1 + \frac{1}{4} + \frac{1}{16} + \cdots .$$

Moreover, the second line becomes exactly $\frac{1}{4}$ times the first, the next $\frac{1}{16}$ times the first, and so on forever. Therefore the whole expression is

$$\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right)^2 = \left(\frac{1}{1-\frac{1}{4}}\right)^2 = \frac{16}{9} .$$

Substituting into the above expression for P_n we see that P_n approaches a value P given by

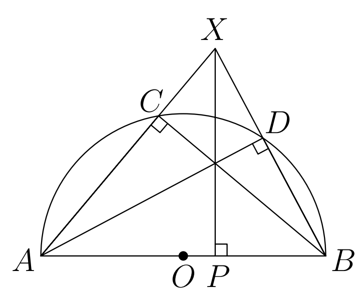
$$4\,\frac{\log P}{\log 4} = \frac{16}{9}$$

and therefore $P = 4^{4/9}$.

Q1624 A semicircle has diameter AB. Two chords AC and BD meet at a point X outside the semicircle. Prove that

$$(AC)(AX) + (BD)(BX) = (AB)^2 .$$

SOLUTION Join BC and AD; let P be the foot of a perpendicular from X to AB. Note that $\angle ACB$ and $\angle ADB$ are angles in a semicircle and therefore are right angles.



Now $\triangle APX$ and $\triangle ACB$ are right–angled triangles with a common angle at A, so they are similar; the same goes for $\triangle BPX$ and $\triangle BDA$. Therefore,

$$\frac{AX}{AB} = \frac{AP}{AC} \quad \text{and} \quad \frac{BX}{AB} = \frac{BP}{BD} ,$$

and so

$$\begin{aligned}(AC)(AX) + (BD)(BX) &= (AP)(AB) + (BP)(AB)\\ &= (AP + BP)(AB)\\ &= (AB)^2 .\end{aligned}$$

Q1625 Find the smallest multiple of 4321 which ends in the digits 1234. *Hint*: just find a suitable multiple first. To investigate whether or not it is the smallest possible, see the article *Linear Diophantine Equations* in *Parabola* Volume 49 Issue 2.

SOLUTION To get a multiple of 4321 in which the last digit is 4, we simply multiply by 4:

$$4321 \times 4 = 17284\,.$$

We want to add another multiple to this to make the second–last digit 3, without changing the last. This means we need to add 5 in the second–last place and 0 in the last; so we add 4321×50. Continue in the same way: as only the last four digits are relevant, we omit the rest.

$$\begin{array}{rr} 4321 \times 4 = & \cdots 7284 \\ 4321 \times 50 = & \cdots 6050 \\ \hline \therefore \quad 4321 \times 54 = & \cdots 3334 \\ 4321 \times 900 = & \cdots 8900 \\ \hline \therefore \quad 4321 \times 954 = & \cdots 2234 \\ 4321 \times 9000 = & \cdots 9000 \\ \hline \therefore \quad 4321 \times 9954 = & \cdots 1234\,. \end{array}$$

Thus 4321×9954 ends in the digits 1234. Is this the smallest? We want a multiple of 4321 which equals a multiple of 10000 plus 1234, that is,

$$4321x = 10000y + 1234\,,$$

where x and y are integers. The theorem on page 6 of the article cited in the question shows that the general solution for x is

$$x = 9954 + 10000t$$

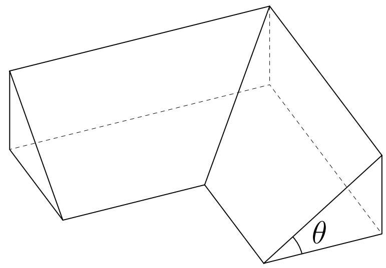
where t is an integer. Now if $t > 0$ this gives a value larger than the one we have already, while if $t < 0$ we have $x < 0$, which is not a valid solution; so the smallest possibility is the one we have already. That is, the smallest multiple of 4321 ending in the digits 1234 is

$$4321 \times 9954 = 43011234\,.$$

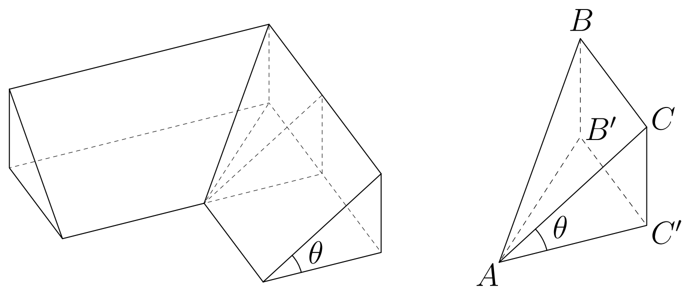
NOW TRY Problem 1637.

Q1626 I want to join at right angles two iron roofs pitched at an angle θ (see the diagram: the upper and lower edges of each roof are to be parallel). At what angles do

I need to cut the iron pieces?



SOLUTION We have added some extra lines to the diagram; and extracted and labelled the important section.



Now $BCC'B'$ is a rectangle, $\angle B'AC' = 45^\circ$ and $\angle AC'B'$ is a right angle, so

$$BC = B'C' = AC' .$$

Also

$$AC' = AC\cos\theta ;$$

so the angle at B is

$$\arctan\left(\frac{AC}{BC}\right) = \arctan\left(\frac{AC'}{AC'\cos\theta}\right) = \arctan(\sec\theta) ,$$

and the angle at A in the original diagram is 180° minus this angle.

Q1627 Find the number of solutions to

$$3x + 2y + z = 2020$$

where x, y and z are positive integers.

SOLUTION Each pair (x, y) with

$$3x + 2y \le 2019 \qquad (*)$$

gives exactly one value of z satisfying the equation. So N, the number of solutions of the equation, is the same as the number of soutions of the inequality $(*)$. If $x = 1$ then $2y \le 2016$; the number of solutions is the number of positive even integers up to 2016, and this is 1008. If $x = 2$, the number of solutions is the number of positive even integers up to 2013, which is 1006. We proceed in the same way until we reach the maximum possible $x = 672$, giving $2y \leq 3$ and just one solution. Hence

$$N = 1008 + 1006 + 1005 + 1003 + \cdots + 6 + 4 + 3 + 1 \,.$$

An easy way to evaluate this sum is to group the terms in pairs and then add up an arithmetic series. The number of solutions is

$$\begin{aligned} N &= (1008 + 1006) + (1005 + 1003) + \cdots + (6 + 4) + (3 + 1) \\ &= 2 \times (1007 + 1004 + \cdots + 5 + 2) \\ &= 1009 \times 336 \\ &= 339024 \,. \end{aligned}$$

Q1628 A path is marked with the integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. A person stands at the point 0 and flips a coin (a fair coin, so that heads and tails come up with probability $\frac{1}{2}$). If the flip is heads, then the person moves one step in the positive direction; if the flip is tails, then he moves one step in the negative direction. This is done repeatedly. Denote by p_n the probability that the person is back at the position 0 after n steps.

(a) Explain why $p_n = 0$ if n is odd.

(b) Suppose that $n = 2k$ is even. Compute $p_n = p_{2k}$.

SOLUTION Since the person moves one step at each toss, they must always occupy even and odd locations alternately. The walker starts at 0, which is even, so after an odd number of tosses, they must be at an odd location and cannot be at 0. This answers (a).

The walker is at 0 after $2k$ tosses if and only if these tosses consisted of k heads and k tails. There are 2^{2k} possible sequences of $2k$ tosses; a sequence satisfying the above condition is found by choosing the k locations for heads out of all $2k$ possible locations, and the number of ways of doing this is $C(2k, k)$. Therefore,

$$p_{2k} = \frac{C(2k, k)}{2^{2k}} \,.$$

Note: $C(2k, k)$ is the binomial coefficient which you may have seen written as ${}^{2k}C_k$ or $\binom{2k}{k}$. That is,

$$C(2k, k) = {}^{2k}C_k = \binom{2k}{k} = \frac{(2k)!}{k!\,k!} \,.$$

NOW TRY Problem 1631.

Q1629 We have a bag and seven slips of paper on which are written

- at least one of the statements in the bag is true;
- at least two of the statements in the bag are true;
- at least three of the statements in the bag are true;
- at least one of the statements in the bag is false;
- at least two of the statements in the bag are false;
- at least three of the statements in the bag are false;
- at least four of the statements in the bag are false.

One of the slips is removed; the other six are placed in the bag and we determine whether they are true or false. How many are false?

SOLUTION For brevity we shall refer to the statements (in the order given) as T1, T2, T3, F1, F2, F3 and F4.

Suppose that there are no false statements in the bag, or only one. Then F2, F3 and F4 are all false and at least two of them are in the bag: this is impossible.

Therefore there are two or more false statements in the bag. This means that F1 and F2 are both true, and at least one of them is in the bag; so T1 is true. Thus we have three true statements, and at least two of them are in the bag; so T2 is true. By a similar argument T3 is true. We now have five true statements; therefore the remaining two (F3 and F4) must both be false and must be in the bag. So **answer**: there are two false statements in the bag.

To be quite sure that this answer is valid (and not self–contradictory), we ought to verify that if any one of F1, F2, T1, T2, T3 is the statement which was removed, then the statements in the bag are true or false as indicated above.

Q1630 Recall from problem 1617 that F is the set of all *dyadic fractions*, that is, fractions in which the denominator is a power of 2; and that for any set X, we write $aX + b$ for the set of all numbers which can be written $ax + b$, where x is in X.

We say that a set is *locally finite* if it has only finitely many elements in any finite interval of the real number line. For example, the set of all integers is not finite, but it is *locally* finite since any interval $p < x < q$ contains only finitely many integers.

(a) Prove that F is not locally finite.

If A and B are sets of numbers, then the *symmetric difference* $A \oplus B$ is the set of all numbers which are in A or B but not both. For example,

$$\{1, 2, 3, 4, 5\} \oplus \{4, 5, 6, 7, 8\} = \{1, 2, 3, 6, 7, 8\} :$$

the numbers $4, 5$ are not included in the right hand side because they are in *both* sets on the left hand side.

We consider subsets X of F with the property

$$(2X) \oplus X \text{ and } (X + \tfrac{1}{2}) \oplus X \text{ are both locally finite.}$$

Since such X are important in a current mathematical research project, we shall call them "important" sets.

(b) Prove that if X is a subset of F (that is, X is some set of dyadic fractions) and X is locally finite, then X is important.

(c) Let X be a subset of F and suppose that $\overline{X}$, the set of all dyadic fractions **not** in X, is locally finite. Show that X is important.

SOLUTION For problem (a), consider an interval $p < x < q$, and let k be a positive integer such that

$$\frac{1}{2^k} < q - p\,.$$

Then for any positive integer n, the interval $2^{n+k}p < x < 2^{n+k}q$ has length greater than 2^n; so there are at least 2^n integers s such that

$$2^{n+k}p < s < 2^{n+k}q\,.$$

This gives 2^n (at least) dyadic fractions satisfying

$$p < \frac{s}{2^{n+k}} < q\,;$$

that is, the interval $p < x < q$ contains at least 2^n elements of F. But n can be chosen as large as we like, so F has infinitely many elements in the interval from p to q, and therefore F is not locally finite.

To solve (b) suppose that X is a locally finite set. We begin by showing that if a, b are fixed dyadic fractions, then $aX + b$ is locally finite. First, if $a = 0$ then $aX + b$ contains the element b and nothing else, and hence is locally finite. Now suppose $a \neq 0$: firstly take $a > 0$, and consider the interval from p to q. The elements of $aX + b$ in this interval satisfy $p < ax + b < q$ and hence

$$\frac{p-b}{a} < x < \frac{q-b}{a}\,;$$

since X is locally finite, there are only finitely many values for x and hence finitely many elements $ax + b$. So $aX + b$ is locally finite. A very similar argument applies if $a < 0$.

Taking $a = 2$, $b = 0$ shows that $2X$ is locally finite. Now any interval of the real line contains finitely many elements of $2X$, finitely many elements of X, and hence finitely many elements of $(2X) \oplus X$; so this set is locally finite. Taking $a = 1$, $b = \frac{1}{2}$, the same argument proves that $(X + \frac{1}{2}) \oplus X$ is locally finite; and this shows that X is important.

For (c), we note two things for a start.

- If a, b are dyadic fractions and a is a power of 2, then $\overline{aX + b} = a\overline{X} + b$. Why is this true? – a number y is in $\overline{aX + b}$ if and only if y is in F and not in $aX + b$; because of Problem 1617(c), this is the same as saying that y is in $aF + b$ and not in $aX + b$; that is, $(y - b)/a$ is in F and not in X; so y is in $a\overline{X} + b$.
- For any sets, we have $A \oplus B = \overline{A} \oplus \overline{B}$: this is because each side consists of the numbers which are in A but not B, together with those which are in B but not A.

Now suppose $\overline{X}$ is locally finite. From (b) we know that $(2\overline{X}) \oplus \overline{X}$ and $(\overline{X} + \frac{1}{2}) \oplus \overline{X}$ are locally finite; the first bullet point above says that $\overline{2X} \oplus \overline{X}$ and $\overline{X + \frac{1}{2}} \oplus \overline{X}$ are locally finite; and the second says that $(2X) \oplus X$ and $(X + \frac{1}{2}) \oplus X$ are locally finite. So X is important.

NOW TRY Problem 1640.