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Infinite products and their applications to infinite series

Sarthak Sahoo[1]

1 Introduction

In mathematics, people have dealt with infinities in different forms, including infinite products. The study of infinite products appeared, among other places, in the 1593 works of François Viète [1, pp. 94–95] on the quadrature of the circle; that is, Riemann sums to find the area of a circle:

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots . \tag{1}$$

An infinite product is more generally written as $a_1 a_2 a_3 \cdots$ where $a_i \in \mathbb{R}$ or, using some useful notation, as

$$P = \prod_{n=1}^{\infty} a_n . \tag{2}$$

A natural question that comes to the mind in the evaluation of the infinite products is whether they converge to some limiting value. For instance, the first of the following three products converges to 0; the second tends to infinity; and the third does not converge to any number:

$$\prod_{n=1}^{\infty} \frac{1}{n} = 1 \times \frac{1}{2} \times \frac{1}{3} \times \cdots \qquad \prod_{n=1}^{\infty} n = 1 \times 2 \times 3 \times \cdots \qquad \prod_{n=1}^{\infty} (-1)^n = 1 \times (-1) \times 1 \times \cdots .$$

Interestingly, we will soon see that there are criteria for when infinite products converge. This article is an introduction to the topic of infinite products and a treatment of some of the very elegant results in this area of mathematics. Leonhard Euler (1707 – 1783) was a pioneer in this area (and in most) areas of mathematics. Below is one of his formulas that expresses the famous Riemann zeta function as an infinite product of primes:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} .$$

This is extremely important in number theory and is related to the Riemann Zeta Conjecture which is one of the million-dollar Millenium Problems posed by the Clay Mathematics Institute.

[1] Sarthak Sahoo is a student in second year studying mathematics honours at UNSW Sydney.

2 Convergence of an infinite product

It turns out that it is useful to write the product (2) as

$$P = \prod_{n=1}^{\infty} (1 + b_n) \,, \tag{3}$$

where $b_n = a_n - 1$ for all $n = 1, 2 \ldots$. By taking the natural logarithm on both sides of the above expression, we turn the product into a sum:

$$\ln(P) = \ln\left(\prod_{n=1}^{\infty}(1+b_n)\right) = \sum_{n=1}^{\infty} \ln(1+b_n) \,.$$

Therefore,

$$P = \exp\left(\sum_{n=1}^{\infty} \ln(1+b_n)\right) .$$

Now we have an infinite sum, we can apply the comparison test to determine convergence. Using calculus, it is not hard to show that

$$\ln(1+x) \leq x \quad \text{for all} \quad x \geq 0 \,.$$

It follows that if $b_n \geq 0$ for all $n \geq 0$, then

$$\prod_{n=1}^{\infty}(1+b_n) = \exp\left(\sum_{n=1}^{\infty}\ln(1+b_n)\right) \leq \exp\left(\sum_{n=1}^{\infty} b_n\right) . \tag{4}$$

Therefore, if

$$\sum_{n=1}^{\infty} b_n$$

converges, then so will

$$P = \prod_{n=1}^{\infty}(1+b_n) \,.$$

However, the converse may not hold.

Moreover, we can prove that the infinite product $P = \prod_{n=1}^{\infty} a_n = \prod_{n=1}^{\infty}(1+b_n)$ is completely dependent on the infinite series $\sum_{n=1}^{\infty} b_n$ for convergence. To do this, we can use the Weierstrass inequality

$$1 + \sum_{n=1}^{\infty} b_n \leq \prod_{n=1}^{\infty}(1+b_n) \tag{5}$$

together with (4) to get the inequalities

$$1 + \sum_{n=1}^{\infty} b_n \leq \prod_{n=1}^{\infty}(1+b_n) \leq \exp\left(\sum_{n=1}^{\infty} b_n\right) . \tag{6}$$

From this, we can see that the product $P = \prod_{n=1}^{\infty} a_n$ converges exactly when the series $\sum_{n=1}^{\infty} b_n$ converges, and we can often use comparison tests to check whether the latter is true.

3 Evaluation of certain products

3.1 Application of the convergence of an infinite product

To apply our observations in the previous section, consider the following example.

$$P_1 = \prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) . \tag{7}$$

Note that if $S = \sum_{n=2}^{\infty} \frac{-1}{n^2}$ converges, then so does P_1. The convergence of S is well known due to Euler: it equals $\frac{-\pi^2}{6}$. Hence, P_1 converges to some number. Let us now find that number. Much like we have telescoping sums in infinite series, we also have telescoping products in infinite products, and this is one of those.

$$P_1 = \prod_{n=2}^{\infty} \frac{(n-1)(n+1)}{n^2} = \left(\frac{1 \times 2 \times 3 \times \cdots}{2 \times 3 \times 4 \times \cdots}\right) \left(\frac{3 \times 4 \times 5 \times \cdots}{2 \times 3 \times 4 \times \cdots}\right) = \frac{1}{2} .$$

Note that these calculations are only justified since the convergence of the infinite product P_1 was established before. We also could have found the value of P_1 by looking at the partial products:

$$\begin{aligned} &\lim_{n\to\infty} \prod_{n=2}^{n} \frac{(n-1)(n+1)}{n^2} \\ &= \lim_{n\to\infty} \left(\frac{1 \times 2 \times \cdots \times (n-1)}{2 \times 3 \times \cdots \times n}\right) \left(\frac{3 \times 4 \times \cdots \times (n+1)}{2 \times 3 \times \cdots \times n}\right) = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} . \end{aligned}$$

3.2 A result of François Viète

Let us show the infinite product (1) due to François Viète, namely

$$\frac{2}{\pi} = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots . \tag{8}$$

Define

$$f(x) = \sqrt{\frac{1+x}{2}}$$

and note that the sequence of terms in the formula (8) has the recurring pattern:

$$\begin{aligned} f(0) &= \sqrt{\frac{1}{2}} \\ f(f(0)) &= \sqrt{\frac{1 + \sqrt{\frac{1}{2}}}{2}} = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \end{aligned}$$

$$f(f(f(0))) = \sqrt{\frac{1+\sqrt{\frac{1+\sqrt{\frac{1}{2}}}{2}}}{2}} = \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}$$

$$\vdots$$

$$f^n(0) = \overbrace{f(f(f(\cdots f(0))))}^{n \text{ times}}$$

$$\vdots$$

The product (8) can therefore be represented as

$$\prod_{n=1}^{\infty} f^n(0)\,.$$

Note that if we set $x = \cos(2\theta)$ and suppose that $0 \leq \theta \leq \frac{\pi}{2}$, then

$$f(\cos(2\theta)) = \sqrt{\frac{1+\cos(2\theta)}{2}} = \sqrt{\cos^2\theta} = \cos\theta\,.$$

We will now use the formula

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \cos\left(\frac{x}{2^n}\right)\,. \tag{9}$$

This formula is quite simple to prove. Indeed, first note that

$$\lim_{y\to 0} \frac{\sin y}{y} = \cos 0 = 1 \tag{10}$$

and calculate:

$$\begin{aligned}
\sin x &= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \\
&= 2\left(2\sin\left(\frac{x}{2^2}\right)\cos\left(\frac{x}{2^2}\right)\right)\cos\left(\frac{x}{2}\right) \\
&= \cdots \\
&= 2^n \sin\left(\frac{x}{2^n}\right)\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)\,.
\end{aligned}$$

By dividing both sides by x and taking the limit $n \to \infty$, the limit (10) gives us (9).

Now evaluate that expression in $x = \frac{\pi}{2}$:

$$\frac{2}{\pi} = \frac{\sin\left(\frac{2}{\pi}\right)}{\left(\frac{2}{\pi}\right)} = \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{16}\right)\cdots\,. \tag{11}$$

We have shown above that $f(\cos(2\theta)) = \cos\theta$, so

$$f(0) = f\left(\cos\left(\frac{2\pi}{4}\right)\right) = \cos\left(\frac{\pi}{4}\right) = \sqrt{\frac{1}{2}}\,.$$

Therefore,

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} \cos\left(\frac{\pi}{2^{n+1}}\right) = \prod_{n=1}^{\infty} f^n(0) = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots .$$

3.3 Differentiation of infinite products and the ratio test

Differentiation of an infinite product would only be meaningful if we know the interval of convergence for our infinite product. Note that if the product

$$P = \prod_{n=1}^{\infty} (a_n) = \exp\left(\sum_{n=1}^{\infty} \ln(a_n)\right)$$

converges, then $\ln(a_n)$ must tend to 0 as n grows, so $\lim_{n\to\infty} a_n = 1$. This implies that

$$\left|\lim_{n\to\infty} \frac{\ln(a_{n+1})}{\ln(a_n)}\right| \le 1\,.$$

One can find the interval of convergence using the above ratio test.

Let us calculate this interval for the product expansion (11) of $\frac{\sin x}{x}$. First note that, by the Chain Rule for differentiation,

$$\frac{d}{dx} \ln\cos x = \frac{\frac{d}{dx}\cos x}{\cos x} = -\frac{\sin x}{\cos x} = -\tan x\,.$$

Therefore, L'Hôpital's rule implies that

$$\lim_{n\to\infty} \left|\frac{\ln\cos\left(\frac{x}{2^{n+1}}\right)}{\ln\cos\left(\frac{x}{2^n}\right)}\right| = \lim_{n\to\infty} \left|\frac{1}{2}\frac{\tan\left(\frac{x}{2^{n+1}}\right)}{\tan\left(\frac{x}{2^n}\right)}\right| \le \frac{1}{2} \le 1\,.$$

Therefore, we have that the product (11) converges, and hence we can differentiate the formula:

$$\frac{d}{dx}\frac{\sin x}{x} = \frac{d}{dx}\left(\exp\left(\sum_{n=1}^{\infty} \ln\cos\left(\frac{x}{2^n}\right)\right)\right)\,.$$

Evaluating each side of the above formula gives

$$\frac{x\cos x - \sin x}{x^2} = -\prod_{n=1}^{\infty} \cos\left(\frac{x}{2^n}\right) \cdot \sum_{n=1}^{\infty} \frac{1}{2^n} \tan\left(\frac{x}{2^n}\right)\,.$$

By (11),

$$\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} = -\frac{\sin(x)}{x}\sum_{n=1}^{\infty}\frac{1}{2^n}\tan\left(\frac{x}{2^n}\right) ,$$

which implies that

$$\frac{1}{x} - \frac{1}{\cot x} = \sum_{n=1}^{\infty}\frac{1}{2^n}\tan\left(\frac{x}{2^n}\right) . \tag{12}$$

4 The Basel Problem

This is one of the problems that I cherish and spent a lot of time to trying to prove, mostly because it's quite an elementary and primary problem in the study of infinite series. The Basel Problem was proposed by Pietro Mengoli in 1650 and solved by Leonhard Euler in 1734 [2]. The problem is to find the closed form of the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2} . \tag{13}$$

To prove the Basel Problem, we begin by establishing the convergence of the series. Since

$$a_n = \frac{1}{n^2}$$

is monotonic decreasing and

$$\int_1^{\infty}\frac{1}{x^2}\,dx = 1 ,$$

the *integral test* implies that the sum (13) converges. To find the value that it converges to, first note that

$$\begin{aligned}
\sin\theta &= 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\\
&= 2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)\\
&= 2\left(2\sin\frac{\theta}{2^2}\sin\left(\frac{2\pi}{2^2}+\frac{\theta}{2^2}\right)\right)\left(2\sin\left(\frac{\pi}{2^2}+\frac{\theta}{2^2}\right)\sin\left(\frac{3\pi}{2^2}+\frac{\theta}{2^2}\right)\right)\\
&= 2^3\sin\left(\frac{\theta}{2^2}\right)\sin\left(\frac{\pi+\theta}{2^2}\right)\sin\left(\frac{2\pi+\theta}{2^2}\right)\sin\left(\frac{3\pi+\theta}{2^2}\right) .
\end{aligned}$$

By continuing this iteration n times, we get

$$\sin\theta = 2^{2^n-1}\sin\left(\frac{\theta}{2^n}\right)\sin\left(\frac{\pi+\theta}{2^n}\right)\sin\left(\frac{2\pi+\theta}{2^n}\right)\sin\left(\frac{3\pi+\theta}{2^n}\right)\cdots\sin\left(\frac{(2^n-1)\pi+\theta}{2^n}\right) .$$

Note that the last two factors above can be written as

$$\sin\left(\frac{2\pi-\theta}{2^n}\right) \qquad \text{and} \qquad \sin\left(\pi-\frac{\pi-\theta}{2^n}\right) = \sin\left(\frac{\pi-\theta}{2^n}\right) .$$

Hence,

$$\sin\theta = 2^{2^n-1}\sin\left(\frac{\theta}{2^n}\right)\left(\sin\left(\frac{\pi+\theta}{2^n}\right)\sin\left(\frac{\pi-\theta}{2^n}\right)\right)\left(\sin\left(\frac{2\pi+\theta}{2^n}\right)\sin\left(\frac{2\pi-\theta}{2^n}\right)\right)\cdots \tag{14}$$

where the last factor is

$$\sin\left(\frac{2^{n-1}\pi+\theta}{2^n}\right) = \sin\left(\frac{\pi}{2}+\frac{\theta}{2^n}\right) = \cos\frac{\theta}{2^n}\,.$$

Now note that

$$\sin\left(\frac{\pi+\theta}{2^n}\right)\sin\left(\frac{\pi-\theta}{2^n}\right) = \sin^2\left(\frac{\pi}{2^n}\right) - \sin^2\left(\frac{\theta}{2^n}\right)\,.$$

Therefore, writing $u = 2^n$, we see that

$$\sin\theta = 2^{u-1}\sin\left(\frac{\theta}{u}\right)\left(\sin^2\left(\frac{\pi}{u}\right)-\sin^2\left(\frac{\theta}{u}\right)\right)\left(\sin^2\left(\frac{2\pi}{u}\right)-\sin^2\left(\frac{\theta}{u}\right)\right)\cdots\,. \tag{15}$$

Dividing (15) on both sides by $\sin\frac{\theta}{u}$ and letting $\theta \to 0$, the limit (10) implies that

$$u = 2^{u-1}\left(\sin^2\left(\frac{\pi}{u}\right)\right)\left(\sin^2\left(\frac{2\pi}{u}\right)\right)\cdots \tag{16}$$

Now divide (15) by (16) to get

$$\sin\theta = u\sin\left(\frac{\theta}{u}\right)\left(1-\frac{\sin^2\left(\frac{\theta}{u}\right)}{\sin^2\left(\frac{\pi}{u}\right)}\right)\left(1-\frac{\sin^2\left(\frac{\theta}{u}\right)}{\sin^2\left(\frac{2\pi}{u}\right)}\right)\cdots$$

Finally, use (10) again, together with the limit $u \to \infty$ and some rearrangement of terms:

$$\frac{\sin\theta}{\theta} = \left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{(2\pi)^2}\right)\cdots \tag{17}$$

This is the famous Euler product representation for $\sin\theta$; see [3]. Now expand (17) and consider the coefficient of θ^2:

$$-\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\cdots\right)\,. \tag{18}$$

The Maclaurin series for $\frac{\sin\theta}{\theta}$ is

$$\frac{\sin\theta}{\theta} = 1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}+\cdots\,.$$

By comparing the coefficient of θ^2 in the above series with the terms in (18), we arrive at

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{3!} = \frac{\pi^2}{6}\,. \tag{19}$$

5 What next?

I sincerely hope you had fun reading this article. The area of infinite products and series is one that relates to Analysis and Number Theory. There are many more beautifully results in this subject; for instance, one could find many products that multiply to the famous mathematical constants like π and e. Then there is also the question of how well can we approximate certain irrational numbers like π, and how fast can we do it; that is, how fast a series converges.

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References

[1] P. Beckmann, *A History of* π (2nd ed.), The Golem Press, Boulder, CO, 1971. ISBN 978-0-88029-418-8.

[2] R. Ayoub, Euler and the zeta function, *Amer. Math. Monthly.* **81** (1974), 1067–1086. doi:10.2307/2319041.

[3] E. Sandifer, *How Euler Did It*, MAA, eulerarchive.maa.org/hedi/HEDI-2004-03.pdf, lasted accessed on 2021-08-14.