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The Black-Scholes model in the context of econophysics

Noah Bergam[1] and **Teodora Kolarov**[2]

1 Abstract

The Nobel Prize-winning Black-Scholes model (BSM) for financial derivatives pricing is inextricably linked to the study of econophysics, where concepts from statistical physics are applied to economic systems. The BSM's connections to this field exist on various levels, some more coincidental than others. By design, the BSM treats the dynamics of stock options as Brownian motion, a phenomenon first discovered in 1827 in the context of random particle movement. However, the fact that the Black-Scholes partial differential equation can be solved using a transformation into the heat equation is more of a mathematical accident. Likewise, the traditional Black-Scholes model's practical inability to predict the behaviour fat-tailed (as opposed to Gaussian) distributions of stock prices may reflect the limitations of the econophysical modelling approach. This paper reviews these three econophysical connections and explores their implications as to the merits of the statistical physics perspective in economics writ large.

2 Introduction

Econophysics is concerned with adapting the mathematical abstractions of physics to economic models and mindsets. The term itself was coined in 1995 by H. Eugene Stanley [6], but its relevance extends well before then, arguably back to Adam Smith's 1776 *Wealth of Nations* [13] and his discussion of an equilibrium-chasing "Invisible Hand" [11]. The subject of econophysics has been approached from many different angles since then, with varying degrees of success. Jan Tinbergen, winner of the Nobel Memorial Prize in Economic Sciences in 1969, developed the landmark gravitational model of international trade. Entropy and information theory have been used to analyse day trading [21]. In 2002, Smith and Foley [14] attempted to put utility maximization theory in terms of thermodynamic formalisms but their methods were in 2003 refuted by McCauley [11]. The list goes on.

There have been many reputed benefits to the physical perspective in economics, including an overall paradigm shift from purely deductive, analytical models to computational, agent-based ones [2].

[1]Noah Bergam is a freshman at Columbia University in the City of New York.

[2]Teodora Kolarov is a freshman at the Georgia Institute of Technology in Atlanta, Georgia.

This paper reviews one of the most influential products of econophysics and mathematical economics: the Black-Scholes model for European put-option pricing. There are three major points of discussion:

- The underlying, physics-related assumptions which help derive the Black-Scholes partial differential equation (PDE).
- The solution to said PDE which involves a transformation into the classical heat equation [10].
- The Black-Scholes model's inability to account for "fat-tailed" distributions which are often observed in natural group dynamics such as human interaction and thus illustrate a certain limitation to econophysics [2].

This paper focuses on the statistics and calculus as well as the general econophysical philosophy which is core to all three observations.

3 Deriving the Black-Scholes model from econophysical principles

3.1 Background and assumptions

Black-Scholes is a pricing model for financial instruments known as options, which are essentially contracts to exchange any underlying security (often a stock) at a predetermined price. Call options give an individual the right to buy a given security, while put options give an individual the right to sell. European call options, with which the Black-Scholes model was originally concerned, can only be exercised at an expiration date that is set by the contract (as opposed to American options, which the buyer can choose when to exercise) [18]. In creating the model in 1973, economists Fischer Black and Myron Scholes wanted to investigate if investors are truly protected, or have the ability to be protected, against huge risks from changes in the asset price before expiry. Their theory of option pricing can be summed up as follows: *Buying a certain amount of the stock is crucial for the long position in the stock to offset the loss in the short call* [3].

This is the key proposition in modern hedge funding, which uses this strategy to limit risks in financial assets. The Black-Scholes PDE demonstrates that investors can create a riskless portfolio by continuously adjusting the proportions of the short and long call. In an efficient or frictionless market, any portfolio with a zero market risk must have an expected rate of return equal to the riskless interest rate [3]. This derivation establishes an equilibrium condition between the expected return on the option and the expected return on the stock, with the final condition relying on the premise of the riskless interest rate [18].

For the Black-Scholes model, we derive a partial differential equation for the price of an European put option. Here are central assumptions of the derivation (see [16]):

(a) The market has a constant riskless interest rate and constant volatility.

(b) There are no transaction costs for the option.

(c) It is possible to buy or sell any number of stocks, including fractional quantities.

(d) The option is European, meaning it can only be exercised at expiration. An investor would not be able to exercise the option prior to this date.

(e) The price of the underlying stock follows a stochastic process.

(f) Stocks are non-dividend paying.

(g) There exists approximately instantaneous trading of the underlying asset.

Note the connections to physics. Some are more obvious than others. Assumption (e) notably leads us to consider financial dynamics as Brownian motion, an idea which was originally applied to particle movement in fluids. By making stock prices continuous, assumption (c) offers a useful analogy between price and physical space (the relevance of this analogy can be witnessed in Section 3.2). Assumption (b) lends itself well to the idea of frictionlessness in the market.

Due to assumption (e), the modelling of stock option valuation is based on stochastic calculus, or Itô calculus, meaning the motion of the asset price is partially random [20]. Many stochastic processes are based on functions which are continuous, but nowhere differentiable. In other words, stochastic processes make for very jagged-looking functions. As we will see in equation (2), stochastic functions generally consist of a non-stochastic component which account for the long-term behaviour of the function, and a stochastic component which accounts for the sharp motions in the short-term [20].

3.2 Derivation

We can write the function of option price as $V(S, t; \sigma, \mu, E, T, r)$. The two independent variables, S and t, make up the asset's primary dimensions, where S is the asset price and t is time. In terms of parameters, σ is the annual volatility (assumed to be constant), μ is the average rate of growth, E is the strike price, T is the maturity time associated with the option, and r as the interest rate (constant) [16].

Essentially, the function $V(S, t)$ represents the long position, which is positive because one *sells* the option, minus the short position, some constant value multiplied by the value of the asset from shorting Δ units of the underlying [3]. If $\Delta < 0$, then we are in fact buying Δ amount of the underlying asset [3]. The idea is to offset the risk of buying the option by buying a certain amount of stock. The Black-Scholes idea is first to find the proportion Δ such that the portfolio becomes non-stochastic in practice. We assume this asset follows the model of a stochastic process, meaning it follows a log-normal random walk. The log-normal distribution is bound by 0 on the lower side, making it perfect for modelling asset prices which cannot take negative values [8]. This

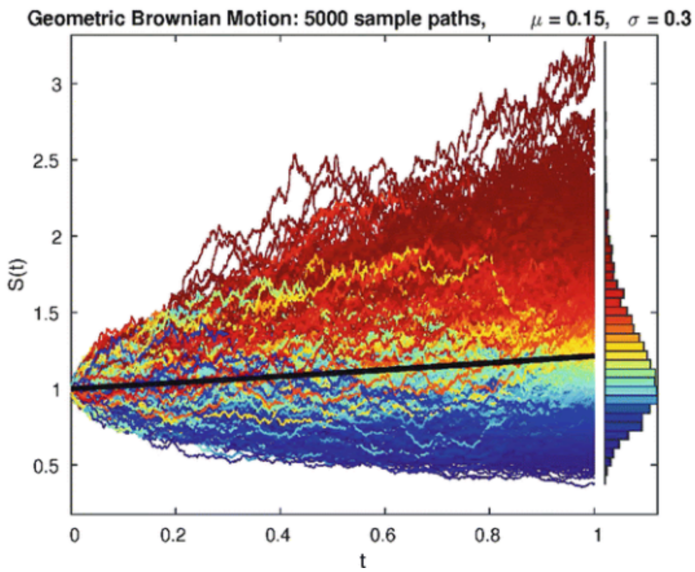

Figure 1: A simulation of Brownian motion with slight upward drift [20].

model describes the movements of stock prices as independent and primarily random events.

Now, we are going to build a portfolio Π of one long option position and a short position in some quantity Δ of the underlying asset S. The quantity Δ will be considered to be a constant. The value of this portfolio is represented as follows (see [8]):

$$\Pi = V(S,t) - \Delta S\,. \tag{1}$$

We can write the valuation of the asset price in a dt time interval by the following differential representation (see [19]):

$$dS = \mu S dt + \sigma S dX(t) dt\,, \tag{2}$$

where $\mu S dt$ is the non-stochastic component, and the change is going to be proportional on the asset price and time interval. Here, $\sigma S dX$ is the stochastic component, composed of unknown volatility σ. The stochastic contribution is given by dX, a function of t.

To explain the function of dX, we consult the Wiener process [16]. A *Wiener process* is defined the following way: The change in X on time interval dt can be expressed as the normal distribution of mean 0 and standard deviation 1 multiplied by the square root of dt. Symbolically, this is expressed as follows:

$$dX = N(0,1)\sqrt{dt}\,, \qquad dX^2 = dt\,. \tag{3}$$

The average of the square of $N(0,1)$ is 1. The average of the square of dX is proportional to dt. These distinctions will be necessary as we analyze the change in the portfolio and keep the linear contributions on dt, despite going up to a second order derivative on the change of the asset [16]. Ultimately, we must analyse the change of the portfolio with respect to the change of the asset on a specific time interval. The change on the value of the portfolio, by linearity, from t to $t + dt$ is

$$d\Pi = (dV - \Delta dS) \,. \tag{4}$$

Let $V = V(S(t), t)$, where S satisfies equation (2). Itô's Lemma [9] tells us that

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial t^2} \right) dt + \sigma \frac{\partial V}{\partial S} dX(t) dt \,. \tag{5}$$

The fundamental difference between stochastic calculus and ordinary calculus is that stochastic calculus allows the derivative to have a random component determined by a Brownian motion [20]. The derivative of a random variable has both a deterministic component and a random component, the latter of which is normally distributed. Itô's Lemma is essentially the chain rule for stochastic calculus [20]. Changes in a variable such as stock price involve a deterministic component which is a function of time and a stochastic component which depends upon a random variable.

From here, we substitute dV into our portfolio equation (4), rearrange, and find that

$$d\Pi = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial t^2} \right) dt + \sigma \frac{\partial V}{\partial S} dX(t) - \Delta\big(\mu S dt + \sigma S dX(t) dt\big) \,. \tag{6}$$

After rearranging variables, we find that

$$d\Pi = \left(\mu S \left(\frac{\partial V}{\partial S} - \Delta \right) + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial t^2} \right) dt + \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) dX \,. \tag{7}$$

In simplifying the expression, we would like to eliminate the random term dX. Since Δ is an arbitrary constant which is meant to fit our needs for equilibrium, we can set $\Delta = \frac{\partial V}{\partial S}$ and simplify to

$$d\Pi = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \,. \tag{8}$$

By making this important distinction, we reduce the randomness, or the risk, to 0, a concept more familiarly known as *delta hedging* [18]. By definition, setting $\Delta = \frac{\partial V}{\partial S}$ delineated the amount of assets that we need to hold to get a riskless hedge [3]. For example, at maturity T, $V(S,T) = \max(S - E, 0)$, a boundary condition that will be elaborated on later. Hence, $\Delta = 1$ if $S > E$ and $\Delta = 0$ if $S < E$. That means we need to hold 1 stock if $S > E$. Meanwhile, if $S < E$, there is no need to hold any stock since there is already a guaranteed profit [3]. The value of Δ is therefore of fundamental importance in both theory and practice.

Our portfolio must also be non-stochastic. Its value has to be the same as it if it were on a bank account with interest rate r. This is known as the fundamental economic assumption of no arbitrage [8]. Investing in the portfolio should be no different than the risk-free alternative. If the change is completely riskless, then it might be the same as the growth we would expect if we put the equivalent amount of cash in a risk-free interest-bearing account [19].

Thus,

$$0 = r\Pi dt - d\Pi\,. \tag{9}$$

The difference in return should be 0, so our conclusion is supported. Now, we substitute in equation (7) and (1), rearrange, and find:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}\,. \tag{10}$$

The left side represents the change in the price of the option V due to time t increasing and the convexity (second derivative of the price of the stocks with respect to interest rates) of the option's value relative to the price of the stock [18]. The right side represents the risk-free return from a long position option and short position with $\frac{\partial V}{\partial S}$ shares of the stock [18].

The differential equation is often represented in terms of Greek letters as follows:

$$\Theta + 1/2\sigma^2 S^2 \Gamma = rV - rS\Delta\,. \tag{11}$$

The key observation of Black and Scholes was that the risk-free return of the portfolio of long and short position options (right hand side) over any time interval was expressed as a weighted sum of Θ and Γ. This is known as the "risk neutral argument". Meanwhile, the value of $\frac{\partial V}{\partial t}$ is most likely negative because the value of an option decreases as it gets closer to its expiration date, and the convexity of the option's value is usually positive from the assumption that the overall portfolio will receive gains from holding the option [19]. Ultimately, the negative from theta and positive from gamma cancel out, resulting in risk-free returns.

Assuming the options contract gives the investor the right to buy the underlying at strike price, E at any time up to and including time T (maturity), then we have the following boundary conditions [19]:

$$V(0,t) = 0 \qquad \text{for all } t\,.$$

If S is ever zero, then dS, the change in asset price, is also zero, and therefore S can never change [3]. Since if $S = 0$ at expiration, then the payoff will be zero, making the option worthless regardless of time T to maturity.

$$V(S,t) \to S \quad \text{as} \quad S \to \infty\,.$$

As $S \to \infty$, the likelihood of the option to be exercised increases, so the strike price becomes less and less important [3]. Thus, the value of the option approaches that of the asset:

$$V(S,T) = \max(S - E, 0)\,.$$

The final condition outlines the payoff at T. This stems directly from the definition of a call option, which will only be exercised if $S > E$, or asset price is greater than the strike price, thereby procuring the gain $S - E$ [8].

4 Connections to the heat equation

4.1 Introduction to the heat equation

The *one-dimensional partial differential heat equation* is as follows:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}\,. \tag{12}$$

where x is position, t is time, and $u(x,t)$ describes the temperature at a point [7]. It is best to imagine this PDE as acting on a thin, thermally conductive rod with varying temperature. The PDE describes a fairly simple principle: that the rate of change of temperature at a given point on the rod is directly proportional to the concavity of temperature as a function of position. In the context of a continuous, wavelike function (see Figure 2), this means that peaks and troughs will change temperature most rapidly.

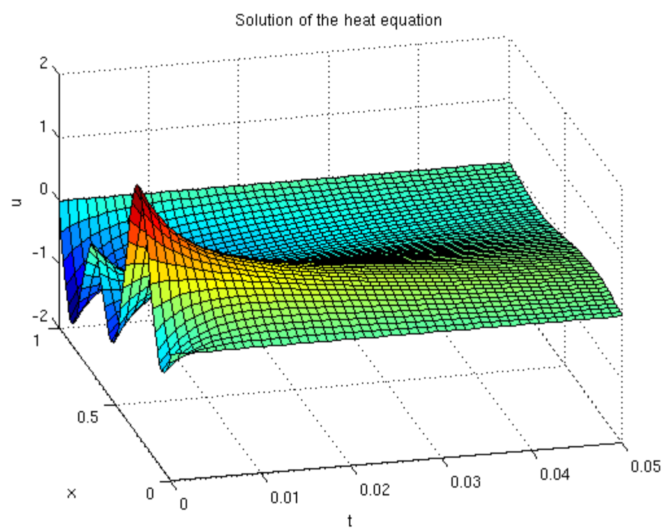

Figure 2: A simulation of the one-dimensional heat equation with initial conditions $u(x,0) = \sin(2\pi x) - \sin(5\pi x)$ and $u(0,t) = 0$. Think of the cross section along the x axis as denoting the temperature along a rod. Note how temperature diffuses over time, and how extrema cool most rapidly [15].

Note that the heat equation can be easily adapted to describe thermodynamic behaviour in higher dimensions:

$$\frac{\partial u}{\partial t} = k\nabla^2 u\,, \tag{13}$$

where u is the heat function, k is a constant, t is time, and the Laplacian operator (∇^2) essentially denotes the second spatial derivative [12]. The same intuition from before applies.

The analytical solution to the Black-Scholes equation involves a transformation into the solvable one-dimensional heat equation. This paper will not provide a solution of the heat equation; we do note, however, that it involves decomposing the function at hand into an infinite sum of sine waves using the Fourier transformation [7].

4.2 Transforming Black-Scholes into the heat equation

Below is the Black-Scholes PDE in its full glory:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0\,. \tag{14}$$

If we think of V as the analogue to temperature, we can already see structural similarities with the heat equation. There is a first derivative of V with respect to time, and a second derivative of V with respect to stock price (analogous to a spatial derivative) [10]. However, there are some notable differences, such as the presence of the $\frac{\partial V}{\partial s}$ and rV. We apply transformations in order to get rid of these terms. See the three subsequent transformations, and their respective purposes, below:

$$\tau = T - t\,. \tag{15}$$

This reverses the progression of time, such that we look forward for our V rather than backwards (since we know the price at maturity, or time T).

$$x = \ln(S) + (r - \frac{1}{2}\sigma^2)\tau\,. \tag{16}$$

This serves to remove the S coefficient in front of the $\frac{\partial^2 V}{\partial S^2}$ term, and get rid of the $\frac{\partial V}{\partial S}$ term altogether.

$$F = Ve^{rt}\,. \tag{17}$$

This brings us to the final heat equation form. The strategy with each of these transformations is to look at every variable on the right hand side that shows up in PDE, compute each necessary partial derivative, substitute, and simplify.

After applying these transformations, we are left with the following equation:

$$\frac{\partial F}{\partial x} = \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}\,. \tag{18}$$

Behold, the heat equation. This can now be solved using Fourier analysis.

5 "Fat Tails": A practical limitation

The past two examples of econophysical connections were based on the Black-Scholes model in theory. This final connection is concerned with a well-established practical limitation of the model: its inability to handle fat-tailed, or leptokurtic, distributions of securities pricing.

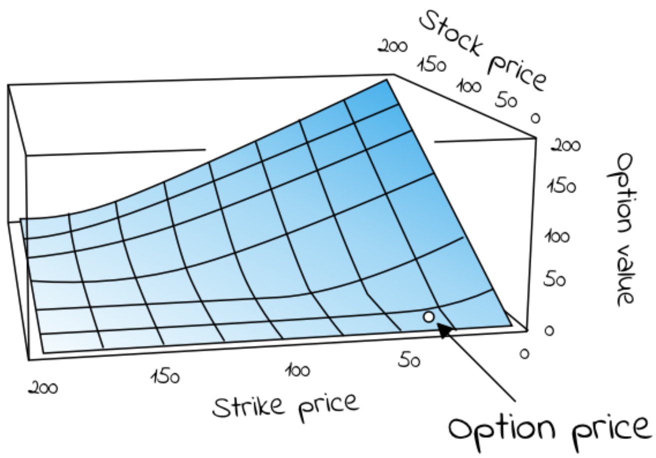

Figure 3: Solution surface of the Black-Scholes equation. Note the similarities to the heat equation surface [18].

One of the fundamental assumptions behind the Black-Scholes model is the idea that securities are normally distributed according to price. However, it has been observed that Black-Scholes overprices call options which are "at the money", such that $S \approx K$. It also underprices call options at either end, whether "in the money", $S \gg K$, or "out of the money", $S \ll K$. This indicates that securities pricing is not normally distributed, but rather follows a more fat-tailed distribution, where extreme events are more likely [5].

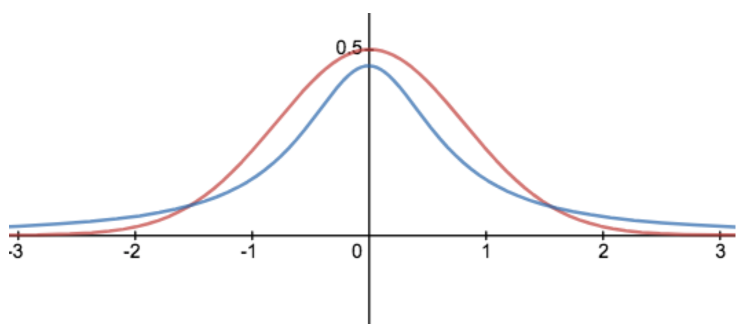

Figure 4: A normal distribution (red) and a more fat-tailed Cauchy distribution (blue).

One of the major contributions of econophysics has been the observation of fat-tailed distributions in financial markets [2]. Note that this does not necessarily demonstrate that fat tails are a common sight in physics and the natural sciences; if anything, fat tails are a more common sight when analysing group dynamics such as human behaviour. Nassim Nicholas Taleb's 2007 book *The Black Swan* [17] discusses this idea in more depth, arguing that leptokurtic distributions are much better models for examining unpredictable human phenomena than the traditional bell curve [4]. From the Black Swan perspective, the traditional Gaussian Black-Scholes model reflects the inherent limitations of the physics perspective in economics and the social sciences in general.

Note that the Black-Scholes model can be and has been adapted to look at fat-tailed distributions, such as Levy Processes and power law distributions [1]. Nonetheless, the question remains: to what extent can a single distribution, no matter how leptokurtic, capture the chaos of collective human behaviour?

6 Conclusions

Econophysics is a powerful but ostensibly limited enterprise. Its pros and cons shine perhaps most clearly through the Black-Scholes partial differential equation. In one sense, it is remarkable how physics-based assumptions of stock motion can yield an analytic (and coincidentally thermodynamics-related) solution to the future value of a call option. The trade-off, at least with the vanilla model of Black-Scholes, is that we are wed to the normal distribution when it appears stock behaviour may follow something more fat-tailed in practice. The philosophical implication, at least at a first glance, is that econophysics has met its match, but judging from the history, this field has a way of adapting.

Adam Smith seemed to believe that the laws that govern economics and the financial market parallel those which regulate the motion of simple harmonic oscillators and falling objects [3]. Neither Smith nor his contemporaries anticipated the sheer volatility of the modern market. Nonetheless, his fledgling concept of econophysics grew from simple mechanics analogies to those of intricate thermodynamics.

Random walk literature arose in the 20th century, asserting that stock price changes are random and cannot be predicted. Cue stochastic, or Itô, calculus, and mathematicians had officially devoted an entire study to differentiation and integration of random variables [20].

In the most recent applications of econophysics, mathematicians have treated frictionless markets as heat baths. The assumption of adequate liquidity, or "market depth" is analogous to the assumption of a heat bath in thermodynamics: Removing a small number of shares from the market does not effect the market price of the asset traded, just as reversibly removing a small amount of energy from the heat bath does not affect overall temperature [11]. With these thermodynamic analogies, it is very possible to draw mathematical relationships with regard to market entropy, which in turn offers insight into a market's sense of disorder [11].

On and on, the field continues to grow in unforeseen and decentralized ways – and yet, on a surface level, there seems to be a certain fundamental absurdity to econophysics. Of course, humans and human transactions don't quite behave as obediently or predictably as particles. But when we zoom out, when we consider the macroeconomic lens, we observe emergent, mathematical properties. The statistics we use to analyse the emergence called an economy can sometimes parallel the statistics used to look at large collections of well-behaved particles. Therein lies the insights, and potentially the folly, of econophysics, in drawing these parallels and chasing them.

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