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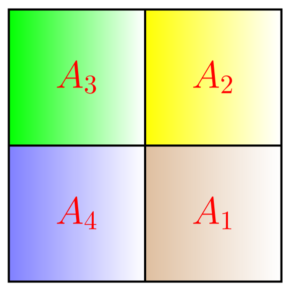
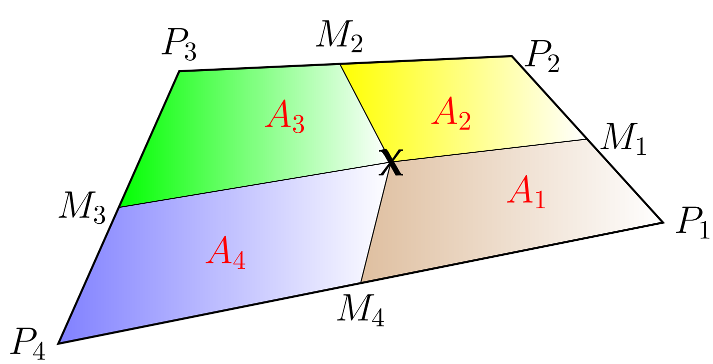
Problems 1651–1660

Problems 1651–1660 are dedicated to the editor of *Parabola*, Thomas Britz, and his partner Ania, in celebration of the arrival of their twin sons Alexander and Benjamin.

Q1651 To celebrate Alexander and Benjamin's 4–month "birthday", Thomas decided to bake a square cake and share it equally among the twins by cutting from the centre of the square to the midpoint of each side, and giving pieces A_1, A_3 to one twin and A_2, A_4 to the other.



Unfortunately, when the cake came out of the oven, it wasn't exactly square. . .

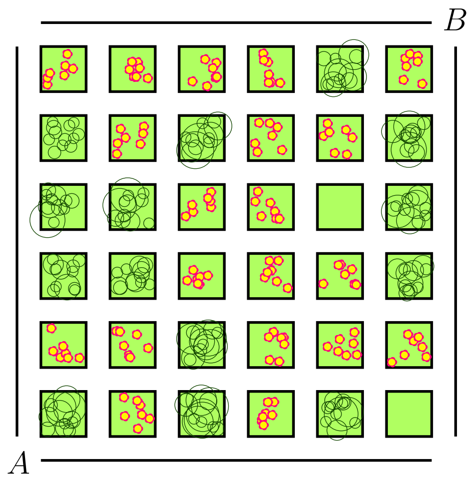


However, the twins' mother Ania was able to assure Thomas that cutting the cake in the same way (from some point to the midpoint of each edge, as before) would still give equal shares to Alexander and Benjamin. Prove that she was correct.

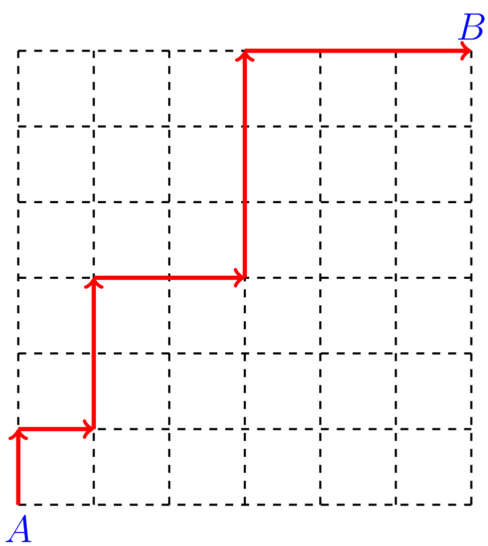
Q1652 Alexander and Benjamin meet a girl called Christine, who tells them that she has a twin sister Denise. What is the probability that Christine and Denise are identical twins?

Assume the following figures: 30% of all twins are identical, 70% are non–identical; identical twin pairs are equally likely to be boy–boy or girl–girl (boy–girl is impossible); of non–identical twins 30% are boy–boy, 30% are girl–girl and 40% are boy–girl.

Q1653 Alexander and Benjamin are playing in their local park. This park consists of an n by n array of square gardens, separated by paths. Alexander starts at the south–west corner of the park and walks along the paths at a speed which takes him along the side of any garden square in exactly one minute, and always heads north or east. Benjamin walks at the same speed, but starts at the north–east corner and always walks south or west. Find the probability that Alexander and Benjamin meet after n minutes.



Q1654 Alexander and Benjamin are walking in the park again, and we can now reveal that in actual fact the number of gardens is 6×6. If Alexander walks from A to B along the path shown in red, in how many ways can Benjamin walk from B to A without meeting Alexander's path (except at the beginning and end of course)? As in the previous problem, Benjamin only walks in a southerly or westerly direction.



Q1655 Alexander and Benjamin are visiting the nation of Twinnia. In this country there is a rule that on any given day, twins must behave alike in terms of telling the truth: that is, both must tell the truth or both must lie; it is forbidden for one to tell the truth and the other to lie. You overhear a conversation between four people. Two of them are Alexander and Benjamin, but you cannot decide which is which, though one of them is wearing a yellow jumper and one is wearing a red jumper. The other two are Ellie and Fiona: they look very similar, and you are not sure whether or not they are twins. The following statements are made.

> Ellie: Fiona and I are twins.
>
> Fiona: the boy in the yellow jumper is Benjamin.
>
> Boy in yellow: the boy in the red jumper is Alexander.
>
> Boy in red: Ellie and Fiona are not twins.

Can you determine which of the boys is which? Can you decide whether Ellie and Fiona are twins or not?

Q1656 Primes which differ by 2 are called **twin primes**. Prove that, with two exceptions, if a and b are twin primes then the last digit of $(a+b)(a^2+b^2)$ is 0.

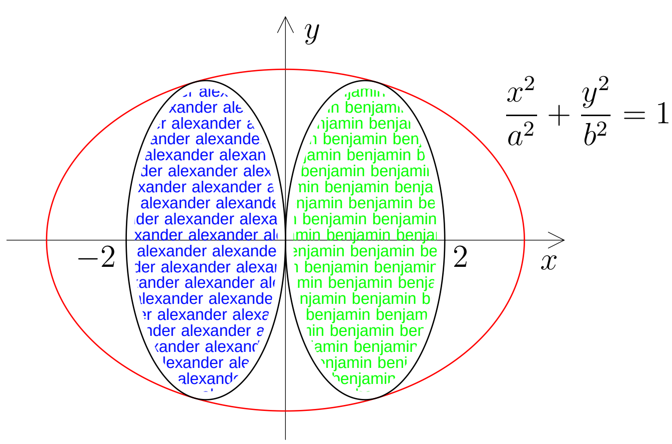
Q1657 Thomas is designing a new nursery for his twins Alexander and Benjamin. They will each have a cradle in the shape of an ellipse, placed side by side. In suitable coordinates, the ellipses have equations

$$(x+1)^2 + \frac{y^2}{4} = 1 \quad \text{and} \quad (x-1)^2 + \frac{y^2}{4} = 1\,.$$

The cradles will be surrounded by a wooden floor. As a mathematician, Thomas is very keen on symmetry, so the surround will also be an ellipse, in this case having equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,;$$

but he is also conscious of using space efficiently, so he wants this ellipse to have the smallest possible area. What values of a and b should he choose?



Q1658 Alexander and Benjamin want to access their favourite computer game. Each has to enter a password, which will be a string of letters a and b. If their words can be converted to each other by substituting aab for bba or vice versa, more than once if necessary, then the game app will agree that the passwords match and will let them access the game. For example, $aaaabab$ and $bbabbba$ match because of the chain of substitutions

$$aa\underline{aab}ab \sim aabb\underline{aab} \sim \underline{aab}bbba \sim bbabbba\,.$$

The twins enter their passwords and hit return... nothing happens! They have made a typing error. Even worse, the backspace/delete keys have frozen!! The only hope is to keep typing and see if the passwords match at some time in the future.

(a) Is this ever possible? That is, are there two non–matching passwords which can be extended to give matching passwords?

(b) Suppose that after realising their mistake, Alexander and Benjamin are very careful to type exactly the same in the future. Now is it possible for them to gain access? In other words, are there two non–matching passwords which can be extended *in the same way* to give matching passwords?

Q1659 Looking ahead a few years... On their first day at school, Alexander and Benjamin are amazed to find that their class consists entirely of twins! – nine pairs of twins, to be exact. The teacher wants to split the class up for three different activities: 7 of the children will do music, 6 will do reading and 5 will do painting. Each pair of twins will do two different activities. In how many ways can the teacher allocate children to activities?

Q1660 Mindful of Alexander and Benjamin's future mathematical education, Thomas assigns a quadratic to each of them:

$$a(n) = 21n^2 + 26n + 8 \quad \text{and} \quad b(n) = 10n^2 + 11n + 3\,.$$

Some time in the future, with careful study and hard work, the twins will be able to show that for any value of the integer n, their expressions $a(n)$ and $b(n)$ will never have any common factor greater than 1. Can you do it now?