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Solutions 1661–1670

Q1661 Do the calculations for Problem 1657 without using calculus. Specifically,

(a) find the gradient of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ at the point (p, q);

(b) find the minimum value of $2a^4 - 2a^3\sqrt{a^2 - 4}$ for $a \geq 2$.

SOLUTION To find the gradient of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (p, q), consider the line

$$y - q = m(x - p)$$

which has gradient m and passes through (p, q). We can eliminate y from these two equations and solve to find the x–coordinates of the points where the line meets the ellipse. In most cases, this will give $x = p$ and one other value. However, if the line is tangent to the ellipse, then we shall obtain $x = p$ only. We need to find the value of m for which this occurs. So, eliminating y and collecting terms in x gives (check for yourself!)

$$(a^2m^2 + b^2)x^2 + 2a^2m(q - mp)x + a^2[(q - mp)^2 - b^2] = 0 \; ;$$

This quadratic equation has a single root if and only if the discriminant is zero:

$$4a^4m^2(q - mp)^2 - 4(a^2m^2 + b^2)a^2[(q - mp)^2 - b^2] = 0 \; .$$

This seems fairly nasty to solve but if you do the algebra carefully, then you will find that many terms cancel and we end up with

$$(p^2 - a^2)m^2 - 2pqm + (q^2 - b^2) = 0 \; . \qquad (*)$$

Now this looks wrong, because it seems to give two possibilities for m, whereas it is geometrically clear that there will be only one tangent to the ellipse at the given point. However, since the point (p, q) is on the ellipse, we have

$$\frac{p^2}{a^2} + \frac{q^2}{b^2} = 1 \; ,$$

which implies

$$p^2b^2 + a^2q^2 = a^2b^2 \quad \text{and so} \quad (p^2 - a^2)(q^2 - b^2) = p^2q^2 \; .$$

Therefore, multiplying the left hand side of $(*)$ by $p^2 - a^2$ gives a perfect square,

$$(p^2 - a^2)^2m^2 - 2(p^2 - a^2)pqm + p^2q^2 = [(p^2 - a^2)m - pq]^2 \; ,$$

and there is one solution

$$m = \frac{pq}{p^2 - a^2} = \frac{pqb^2}{p^2b^2 - a^2b^2} = -\frac{pqb^2}{a^2q^2} = -\frac{b^2}{a^2}\,\frac{p}{q}\,,$$

as we found in Problem 1657.

Secondly, we want to find the minimum possible value of

$$f(a) = 2a^4 - 2a^3\sqrt{a^2 - 4}$$

if $a \geq 2$. We can substitute $a = \frac{2}{\cos\theta}$ with $0 \leq \theta < \frac{\pi}{2}$: this will remove the square root term to give

$$f(a) = \frac{32}{\cos^4\theta} - \frac{32\tan\theta}{\cos^3\theta} = 32\frac{1 - \sin\theta}{\cos^4\theta}\,.$$

Next, eliminate the $\frac{1}{\cos\theta}$ terms by writing $f(a)$ as

$$f(a) = 32\frac{1 - \sin\theta}{(1 - \sin^2\theta)^2} = \frac{32}{(1 + t)(1 - t^2)}\,,$$

where $t = \sin\theta$. To find the minimum value of $f(a)$ we want the maximum value of $(1 + t)(1 - t^2)$ for $0 \leq t < 1$. This will be a value c for which the horizontal line $t = c$ is tangent to the graph of $(1+t)(1-t^2)$, and for the same kind of reason as in the previous problem, we need a value of c such that the equation $(1 + t)(1 - t^2) = c$ has a double root. This means we require

$$t^3 + t^2 - t + (c - 1) = (t - \alpha)^2(t - \beta)$$

for some α, β; then α will be the relevant t value (which must lie between 0 and 1), and c will be the maximum we are looking for. Expanding and equating coefficients in the above cubic equation gives

$$1 = -2\alpha - \beta\,, \quad -1 = 2\alpha\beta + \alpha^2\,, \quad c - 1 = -\alpha^2\beta\,.$$

Solving the first equation for β and substituting into the second gives a quadratic equation with solutions $\alpha = \frac{1}{3}, -1$; since α lies between 0 and 1, the latter possibility must be rejected. This gives $\beta = -\frac{5}{3}$ and $c = \frac{32}{27}$, and so the minimum value we seek is

$$f(a) = \frac{32}{32/27} = 27\,.$$

The value of a that attains this minimum is

$$a = \frac{2}{\cos\theta} = \sqrt{\frac{4}{\cos^2\theta}} = \sqrt{\frac{4}{1 - \sin^2\theta}} = \sqrt{\frac{4}{1 - t^2}} = \sqrt{\frac{4}{1 - \alpha^2}} = \sqrt{\frac{4}{1 - \frac{1}{3^2}}} = \frac{3}{\sqrt{2}}\,,$$

as we found previously.

Q1662 This is a variation of Problem 1659. Now we have eight pairs of twins, and there are four activities, music, painting, reading and dancing, with four children to do each activity. Once again, each pair of twins is to do two separate activities. In how many ways can children be allocated to activities?

SOLUTION
Step 1: choose four pairs of twins to do music: this can be done in $C(8,4)$ ways.

Step 2: suppose that k of the "non–music" pairs do painting; then $4-k$ of the "music pairs" also do painting, and the number of ways to choose the "painting pairs" is $C(4,k)C(4,4-k)$. This leaves $4-k$ pairs with no activity as yet: they must do reading and dancing. There remain $2k$ pairs with only one activity so far: k of them will be chosen to take the k remaining reading spots, and the other k must do dancing: there are $C(2k,k)$ ways to make this choice. This gives

$$C(4,k)C(4-k)C(2k,k)$$

options for step 2; however, k could be any number from 0 to 4, so we need to sum this expression over all values of k. Simplifying by noting that $C(4,4-k) = C(4,k)$, the total number of options is

$$C(4,0)^2C(0,0) + C(4,1)^2C(2,1) + C(4,2)^2C(4,2) \\ + C(4,3)^2C(6,3) + C(4,4)^2C(8,4) = 639\ .$$

Step 3: choose which twin in each pair does which activity: there are 2^8 choices. Putting all this back together, we get the total number of allocations

$$C(8,4) \times 639 \times 2^8 = 11450880\ .$$

Q1663 As in Problem 1643 (*Parabola* Volume 57, Issue 1), a positive integer with k digits $d_0d_1\cdots d_{k-1}$ in base 10 is called a *Geezer number* if the digits consist of exactly d_0 zeros, exactly d_1 ones, exactly d_2 twos and so on. Show that in a k–digit Geezer number, at most one of the digits $d_3, d_4, \ldots, d_{k-1}$ is non–zero.

SOLUTION We know from the solution of the previous problem that the digits in question can only be 0 or 1; suppose that two of them are 1, specifically, $d_i = d_j = 1$, where $3 \le i < j$. Then the digits of n include an i and a j; so there is a digit a occurring i times and a digit b occurring j times. If $a \ge 3$ then a might be i or j, but not both; so the sum of the digits of n is

$$S \ge i(a) + i \ge 12\ .$$

This is impossible, so $a \le 2$; and for the same reason, $b \le 2$. Let m be the larger of a and b. The digit m occurs in n at least 3 times; so there are three numbers which occur m times. Now these three numbers occur at most twice, so none of them is a or b (which occur at least three times); two of them could be i and j, but that still leaves at least another number c occurring m times. So the digits include a occurring i times, b occurring j times, c occurring m times and i, j occurring at least once each; therefore

$$\begin{aligned} S &\geq i(a) + j(b) + m(c) + i + j \\ &\geq i(a+b) + c + i + j \\ &= (i-1)(a+b) + (a+b+c) + i + j \\ &\geq 2 + 3 + 3 + 4 \\ &= 12 \end{aligned}$$

which is impossible. So it is impossible that two or more of the digits $d_3, d_4, \ldots$ are non–zero.

NOW TRY Problem 1671.

Q1664 Let a, b, c, d be four prime numbers greater than 5; suppose that $a < b < c < d < a + 10$. Prove that 60 is a factor of $a + b + c + d$ but 120 is not.

SOLUTION If a prime number (other than $2, 3$ or 5) is divided by 30, then the remainder cannot be a multiple of $2, 3$ or 5 and therefore must be one of the numbers

$$1\ ,\ 7\ ,\ 11\ ,\ 13\ ,\ 17\ ,\ 19\ ,\ 23\ ,\ 29\ .$$

We tabulate these numbers together with the distance from each number to the third subsequent number in the list, noting that to do so for the later numbers, we have to extend the above list into the next cycle of 30 numbers.

1	7	11	13	17	19	23	29	31	37	41
12	10	8	10	12	12	14	12			

It is clear that the only way to get four primes with difference between first and last less than 10 is for the first to have remainder 11 when divided by 30. Therefore, for some integer k we have

$$\begin{aligned} a + b + c + d &= (30k + 11) + (30k + 13) + (30k + 17) + (30k + 19) \\ &= 120k + 60\ , \end{aligned}$$

and as required, this is a multiple of 60 but not of 120.

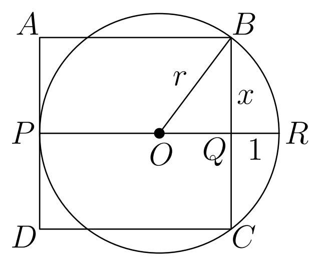
COMMENT. The obvious example of four primes like this is 11, 13, 17, 19. The next is 101, 103, 107, 109. The first example above one million is

$$1002341\ ,\ 1002343\ ,\ 1002347\ ,\ 1002349\ .$$

It is unknown whether or not there are infinitely many examples.

Q1665 Let $ABCD$ be a square, and let P and Q be the midpoints of AD and BC respectively. Suppose that PQR is the diameter of a circle passing through B and C. If $QR = 1$, find the radius of the circle.

SOLUTION Let O be the centre of the circle; let r be the radius and $x = BQ$.



Then $BC = PQ$ and OQB is a right–angled triangle; hence

$$2x = 2r - 1 \quad \text{and} \quad r^2 = x^2 + (r-1)^2 .$$

Solving gives $x = 2$ and hence $r = \frac{5}{2}$.

Q1666 The equation

$$\cos x + \sin y = \cos y + \sin x$$

is obviously true when $\cos x = \cos y$ and $\sin x = \sin y$. Does it have any other solutions?

SOLUTION We rearrange the equation and use the trigonometric formula

$$\cos x - \cos y = \sin x - \sin y$$

to get

$$-2\sin\frac{x+y}{2}\sin\frac{x-y}{2} = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2} . \qquad (*)$$

This is true when $\sin\frac{x-y}{2} = 0$ or, in other words, $x - y = 2n\pi$, where n is an integer; this implies $\cos x = \cos y$ and $\sin x = \sin y$, which we don't want. However, $(*)$ is also true when

$$-\sin\frac{x+y}{2} = \cos\frac{x+y}{2}$$

or, in other words,

$$\tan\frac{x+y}{2} = -1 .$$

This is equivalent to $\frac{x+y}{2} = -\frac{\pi}{4} + n\pi$ and thus to

$$\cos x = \cos\Bigl(-\frac{\pi}{2} - y\Bigr) \quad \text{and} \quad \sin x = \sin\Bigl(-\frac{\pi}{2} - y\Bigr) ,$$

which simplifies to

$$\cos x = -\sin y \quad \text{and} \quad \sin x = -\cos y .$$

These are the only solutions of the equation other than the obvious ones mentioned in the question.

Q1667 A bag contains 8 balls, two each of 4 different colours. They are to be drawn from the bag in a random order and placed in a row.

(a) Find the probability that the row consists of pairs of the same colour, for example, red, red, black, black, blue, blue, white, white.

(b) Find the probability that the second half of the row has the same colours in the same order as the first half, for example, black, blue, white, red, black, blue, white, red.

SOLUTION The probability that the second ball is the same as the first is $\frac{1}{7}$; then the third ball is certain to be of a different colour; the probability that the fourth ball is the same colour as the third is $\frac{1}{5}$; and the probability that the sixth is the same as the fifth is $\frac{1}{3}$. So the probability in (a) is

$$\frac{1}{7}\frac{1}{5}\frac{1}{3} = \frac{1}{105}\,.$$

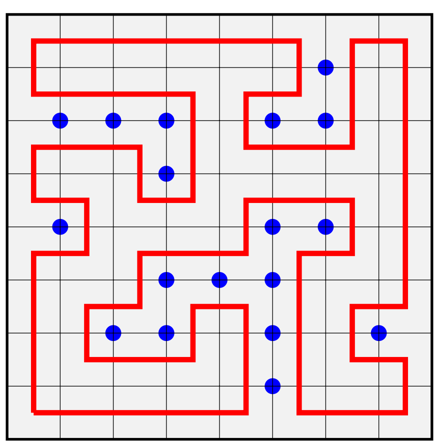
For (b), the first four balls must have different colours: otherwise, since there are only two of each colour, the second four balls cannot have the same colours as the first. So, the probability that the second ball has a different colour from the first is $\frac{6}{7}$; the probability that the third is different from both is $\frac{4}{6}$; the probability that the fourth is different again is $\frac{2}{5}$. Then the remaining four balls must have the same colours as the first four, and the probability that they occur in the same order is $\frac{1}{4!}$. So the probability for (b) is

$$\frac{6}{7}\frac{4}{6}\frac{2}{5}\frac{1}{4!} = \frac{1}{105}\,.$$

That is, the two probabilities are the same.

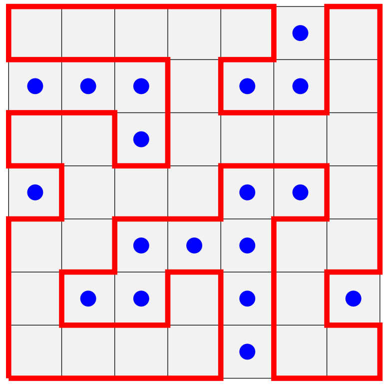
NOW TRY Problem 1672.

Q1668 A closed path consists of lines from the centre of a square to the centre of an adjacent square on a $2n$ by $2n$ grid. The curve visits every square exactly once. An example is shown in the diagram.



There are a number of intersections of gridlines outside the path, shown as blue dots in the diagram. How many?

SOLUTION Interchange squares and gridlines so that the path follows the lines, as in the following diagram.



We may assume that the side length of the squares is 1. The path is now a lattice polygon, that is, a polygon in which every vertex has integer coordinates, and its area is given by **Pick's formula**:

$$A = I + \tfrac{1}{2}B - 1 ,$$

where I is the number of lattice points inside the polygon and B is the number on the boundary. Since the curve visits every square in the original diagram, there are no complete squares inside the path, and therefore no lattice points inside the polygon in the second diagram: that is, $I = 0$. For the same reason, every lattice point is on the boundary, and so $B = (2n)^2$: therefore

$$A = 2n^2 - 1 .$$

Moreover, the total grey area in the second diagram is $(2n-1)^2$: so the grey area outside the polygon is

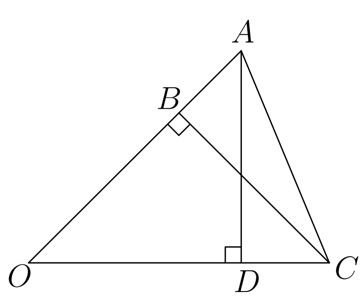
$$(2n-1)^2 - A = 2n^2 - 4n + 2 = 2(n-1)^2 ,$$

and this is also the number of blue dots, and the number of gridline intersections in the original diagram.

As a check, the example given in the question has $n = 4$, so we predict 18 intersections outside the path, and we confirm this simply by counting the dots. This problem was inspired by the "Masyu" puzzle.

Q1669 Let OBC and ODA be right–angled isosceles triangles of the same size such that B and D are the right angles, the point B lies on OA and the point D lies on OC. Use this diagram to evaluate $\tan(\pi/8)$.

SOLUTION



Triangles ABC and CDA are congruent, so angles $\angle ACB$ and $\angle CAD$ are equal. From triangle ABC, therefore, we have

$$\angle ACB = \frac{1}{2}\Big(\pi - \frac{\pi}{2} - \frac{\pi}{4}\Big) = \frac{\pi}{8} \,.$$

And hence

$$\tan\frac{\pi}{8} = \frac{AB}{BC} = \frac{OA}{BC} - \frac{OB}{BC} = \sqrt{2} - 1 \,.$$

Q1670

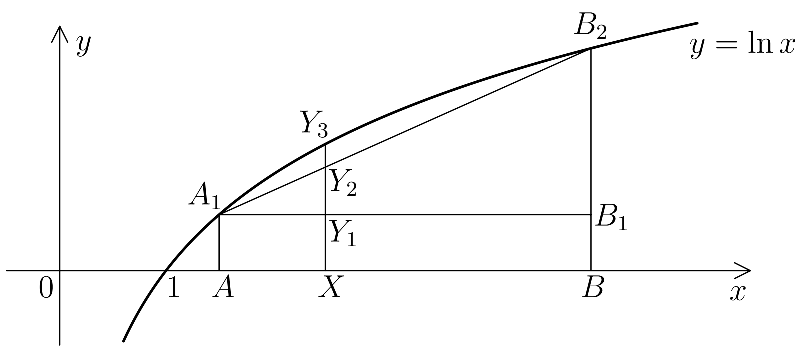
(a) Prove the *weighted arithmetic–geometric mean inequality*:
if a, b are positive numbers and $0 \le p \le 1$, then

$$a^p b^{1-p} \le pa + (1-p)b \,.$$

(b) Use (a) to show that, if $0 \le x \le \frac{\pi}{2}$, then

$$(\cos^2 x)^{\cos^2 x} + (\cos^2 x)^{\sin^2 x} + (\sin^2 x)^{\cos^2 x} + (\sin^2 x)^{\sin^2 x} \le 3 \,.$$

SOLUTION (a) Consider the graph of $y = \ln x$, noting that it is concave downwards.



Let a and b be positive; by symmetry we may assume that $a \le b$. Let $0 \le p \le 1$ and write $x = pa + (1-p)b$; it is easy to see that $a \le x \le b$. Label points as shown in the diagram, where $A = (a, 0)$ and $X = (x, 0)$ and $B = (b, 0)$. Then

$$\frac{AX}{AB} = \frac{(pa + (1-p)b) - a}{b - a} = 1 - p \,;$$

since $\triangle A_1Y_1Y_2$ and $\triangle A_1B_1B_2$ are similar, we have also

$$\frac{Y_1Y_2}{B_1B_2} = 1 - p \,.$$

Therefore,

$$\begin{aligned} XY_2 &= XY_1 + Y_1Y_2 \\ &= \ln a + (1-p)(\ln b - \ln a) \\ &= p\ln a + (1-p)\ln b \,. \end{aligned}$$

It is clear from the concavity of the graph that $XY_2 \leq XY_3$; that is,

$$p \ln a + (1-p) \ln b \leq \ln x = \ln(pa + (1-p)b) \ .$$

Taking e to the power of each side and using well–known logarithm laws to simplify yields

$$a^p b^{1-p} \leq pa + (1-p)b$$

as required.

To prove (b), we begin by considering the above inequality in the case $a = \cos^2 x$, $b = 1$ and $p = \cos^2 x$. Since a, b are positive and $0 \leq p \leq 1$, we have

$$(\cos^2 x)^{\cos^2 x} \leq \cos^4 x + \sin^2 x \ .$$

By similar arguments, we obtain

$$\begin{aligned}
(\cos^2 x)^{\sin^2 x} &\leq \sin^2 x \cos^2 x + \cos^2 x \\
(\sin^2 x)^{\cos^2 x} &\leq \sin^2 x \cos^2 x + \sin^2 x \\
(\sin^2 x)^{\sin^2 x} &\leq \sin^4 x + \cos^2 x \ .
\end{aligned}$$

Finally, adding these four inequalities and using $\cos^2 x + \sin^2 x = 1$ gives

$$\begin{aligned}
&(\sin^2 x)^{\cos^2 x} + (\cos^2 x)^{\sin^2 x} + (\sin^2 x)^{\sin^2 x} + (\cos^2 x)^{\cos^2 x} \\
&\qquad \leq \cos^4 x + 2 \sin^2 x \cos^2 x + \sin^4 x + 2 \cos^2 x + 2 \sin^2 x \\
&\qquad = (\cos^2 x + \sin^2 x)^2 + 2 \\
&\qquad = 3 \ .
\end{aligned}$$