Parabola Volume 58, Issue 1 (2022)

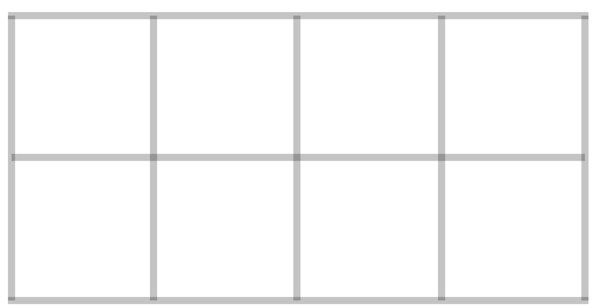
How to draw knots on grids and graphs

Catherine Greenhill[1]

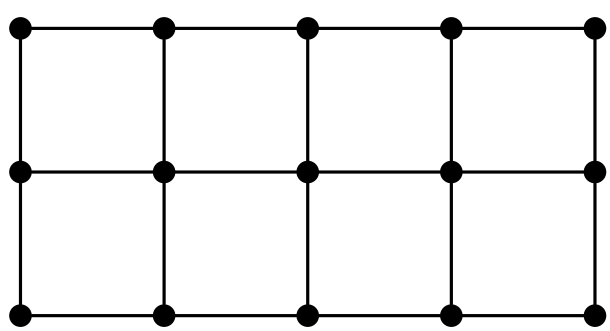
Knotwork is a very mathematical form of decoration which has been used in many cultures, including Roman mosaics, Islamic art, Celtic art and Ethiopian art [2]. In this article, I will show you the basic technique and discuss a couple of mathematical points along the way.

1 Drawing knots on grids

The easiest way to start is with a grid, like the one shown below. We call this a 3×5 grid as it has 3 horizontal lines and 5 vertical lines.



A grid can also be thought of as a graph, or network. A graph has a set of vertices, usually drawn as a black dot, with some pairs of vertices joined by edges. We can draw the 3×5 grid as a graph with 15 vertices and 22 edges.



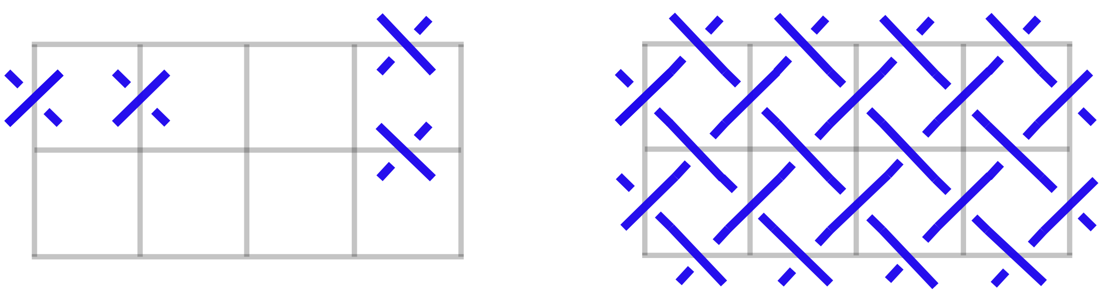
[1]Catherine Greenhill is a Professor at UNSW Sydney.

When drawing a knot on a grid, the first step is to add a *cross* onto each edge. The cross has an *upper* strand and a *lower* strand: the lower strand is drawn with a gap in it.

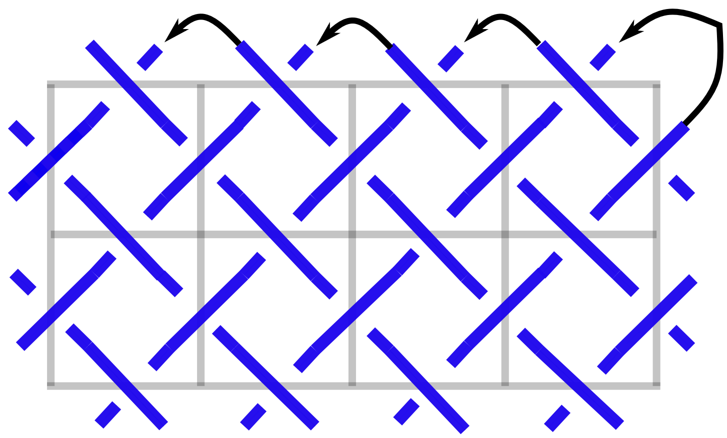
It is very important to make sure that all your crosses have the same orientation. This will make sure that your strands alternate so that as you trace around your design, the lower strand becomes the upper strand at the next crossing. Your knot should always alternate between under and over, like a basket weave. Let's see how to achieve this.

Think of a compass with N for north, E for east, S for south and W for west. If the edge runs N-S, then the *upper* strand goes between NE and SW, while the *lower* strand goes between NW and SE. This takes care of the vertical edges, while for the horizontal edges we just rotate everything by $90°$.

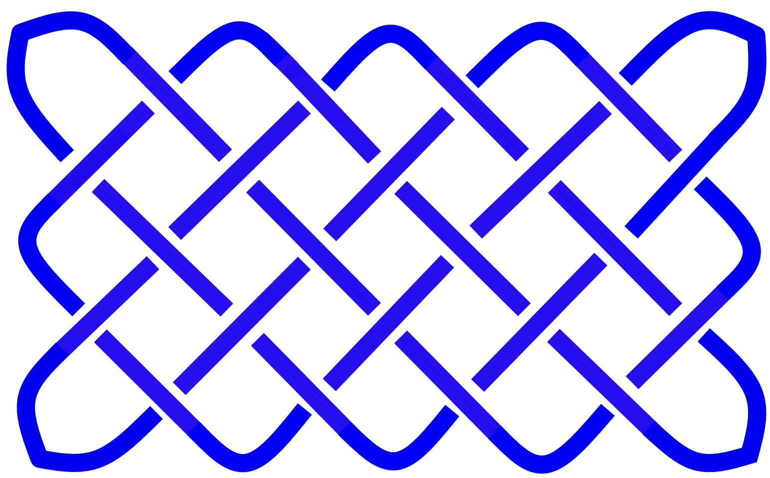
Work through every edge on your grid, adding a cross as shown. In the interior of the grid, you will see that you can easily join your crosses together, giving the picture shown below on the right.



But what do we do with the strands that lie on the edges of the grid? The trick here is to look at each "over" strand in turn. It is pointing in a particular direction. Start walking in that direction, keeping the grey "wall" of the grid on your left. Stop when you find the next "under" strand: this is where the "over" strand must go.



(This rule also works for the interior strands, by the way!) Once you have joined every “over” strand to the next “under” strand, using the rule described above, your drawing should look like this.



We haven’t shown the grid lines here because once your drawing is complete, you don’t need them. If you have drawn the grid lines in pencil, then you can erase them once you are finished.

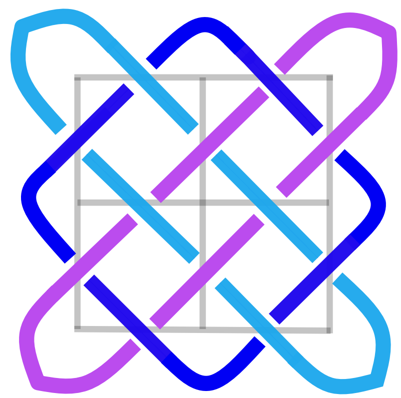
Summary: the basic method uses these steps:

(a) Put a cross on each edge, making sure to use a consistent orientation.

(b) For each “over” strand, find the next “under” strand (keeping the grey edge to your left as you look) and join them up.

Don’t worry if you make a mistake! It just takes some practice.

You can check that the knot we drew on the 3×5 grid consists of one single closed loop. But if you start with a grid of a different shape, then you might end up with more than one closed loop, or *component*. For example, a 3×3 grid leads to a knot with 3 components, shown here with 3 different colours.

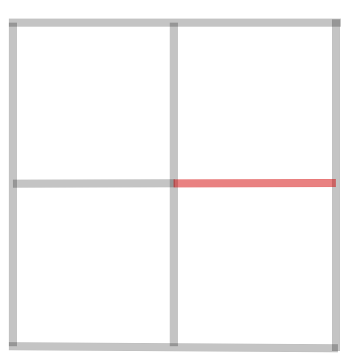


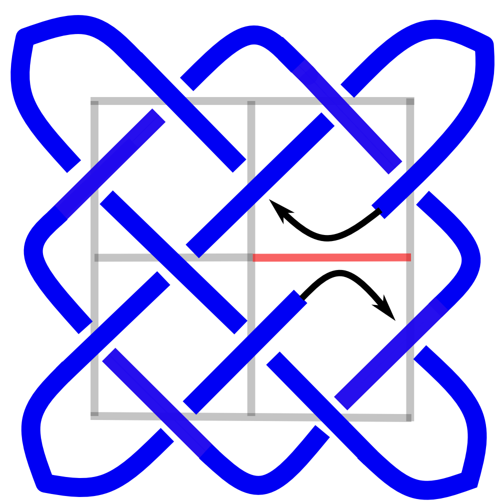
Question: Can you find a relationship between the dimensions of the grid and the number of components[2] in the corresponding knot? In other words, if you start with an $m \times n$ grid, then how many components do you think you will end up with? Can you make a hypothesis? Can you prove it?

2 Symmetry breaking in two ways

Let's look at two ways that you can introduce some variety to your design, still based on a grid. The first way is to mark one or more edges on your grid as impenetrable. These edges become a *barrier*, where strands do not cross. Instead, the barrier reflects the strand back, one on each side.

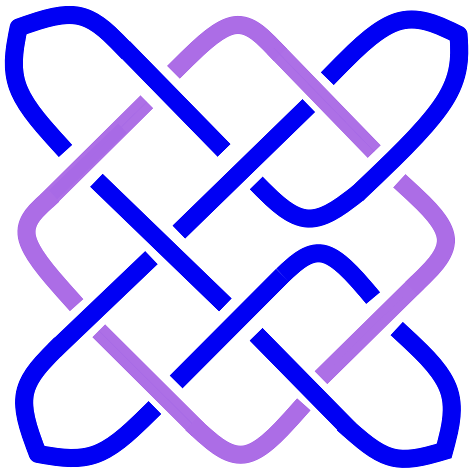
Suppose we take a 3×3 grid and turn one of the interior edges into a barrier (shown in red). After putting a cross on all the other edges (the grey edges), we just need to decide where each "over" strand needs to go. The "over" strands which lie on the edge of the grid are handled as described earlier, leaving two "over" strands which are reflected at the red barrier, as shown on the right.



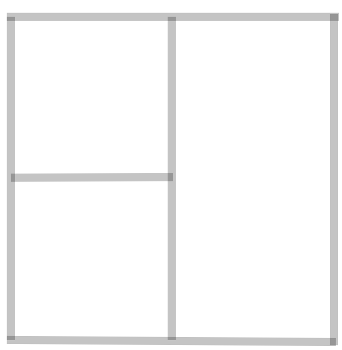


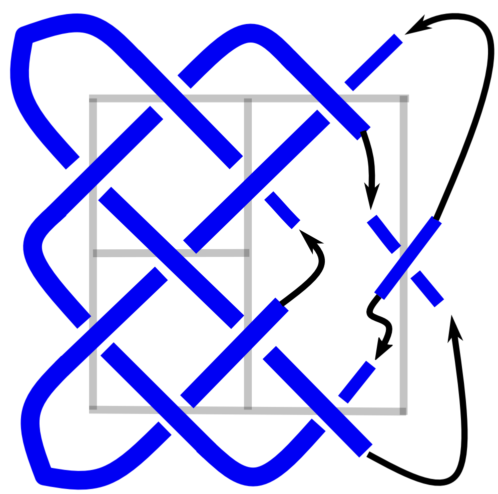
[2]In the mathematical area of knot theory, these drawing would be called "links", and a "knot" would refer to a link with exactly one component. But it is easier to refer to all of our drawings as knots.

We end up with the following knot, which has two components. Adding the barrier has reduced the number of components from 3 to 2, compared with the standard 3×3 grid.

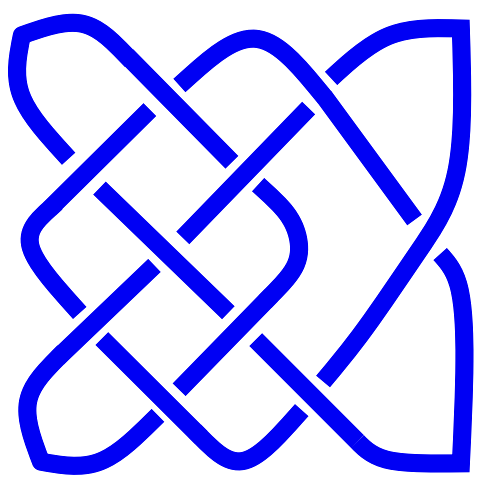


Rather than turn an edge into a barrier, another option is to simply remove that edge. Where the removed edge used to be, there will be no crossing, but there will also be no reflecting barrier. Instead, each "over" strand which enters this space will follow the same rule as for the edge of the grid: walk around, keeping the grey edge on the left, looking for the next "under" strand to connect with. Notice also that now there is only one long vertical edge on the right, so there is only one crossing here.



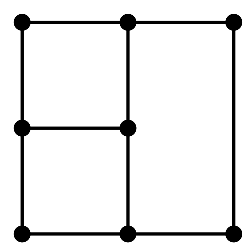


Once you have worked out where each "over" strand connects up, you may have to smooth things out a little before making your final drawing of the knot. This is because when you put a cross on an edge, the angle of the two strands might not match up with where you need that strand to go. After this smoothing step, you might end up with something like this. (The "pointy" corners are traditional: you can use round corners if you prefer.) How many components does this knot have?

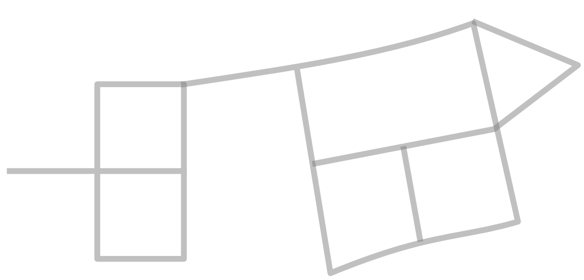
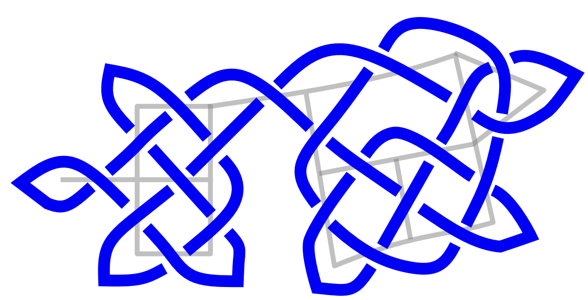


3 Going off grid

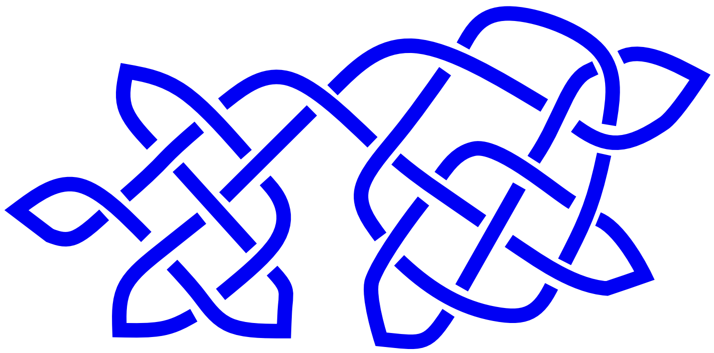
Now you know these techniques, you are ready to abandon the grid. Earlier I mentioned that a grid can be thought of as a graph. In fact, a grid is a *planar graph*, which is a graph that can be drawn on the page with no edges crossing. In our last example, we took a 3×3 grid and deleted one edge. This is the same as applying our technique to the following graph.



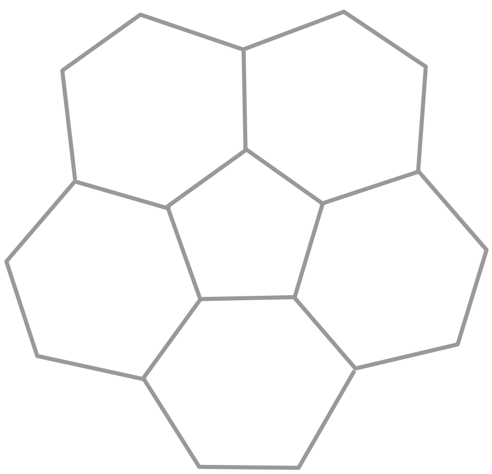
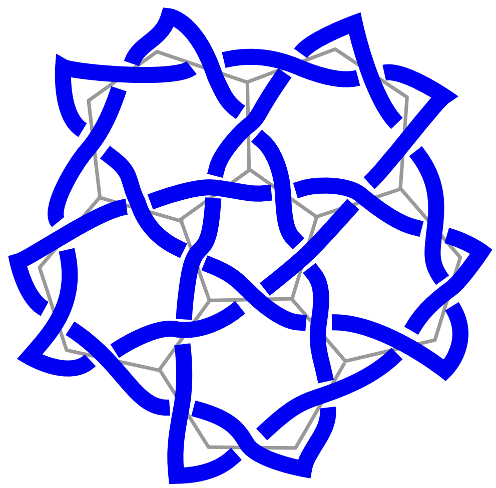
In fact, you can apply the same steps to create knotwork from any planar graph. Just put a cross on each edge, with consistent orientation, and then join each "over" strand to the next "under" strand, using our rule (walk around keeping the edge to the left). Here's one example, which is still a bit grid-like:

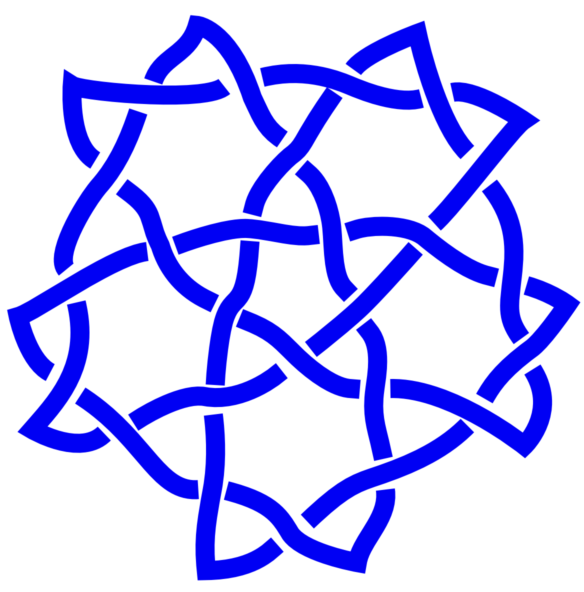
Removing the graph shows the finished knot. How many components does it have?



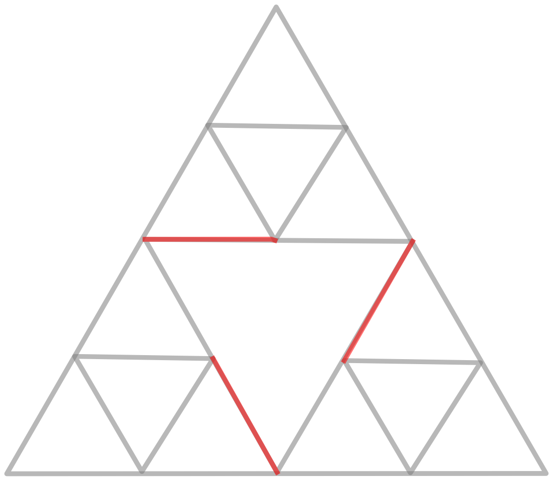
The next design starts with a planar graph made from a pentagon surrounded by six hexagons (like a football).

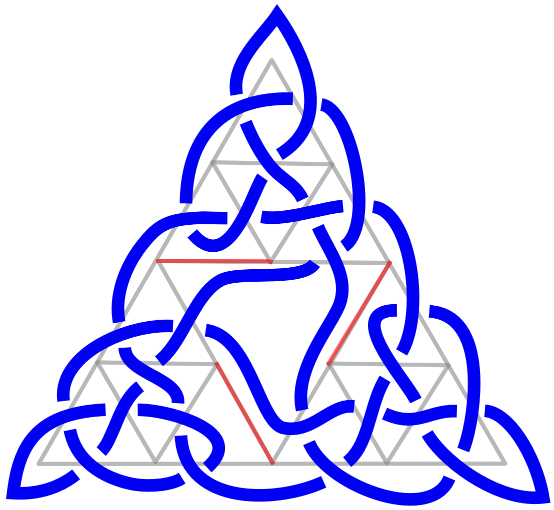
 

Around each "inner face" of the graph (each pentagon or hexagon), the knot looks like a star:

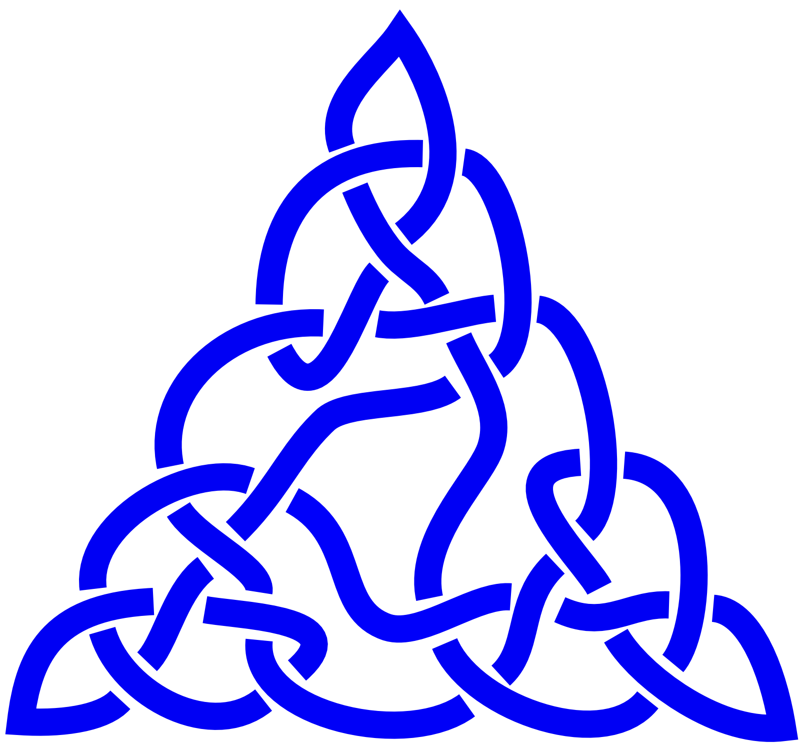


And finally, to make the design shown at the very start of this article, take an early iteration of the Sierpinski triangle and replace 3 of the edges with barriers (shown in red).





The finished product in all its glory:



If you enjoyed this article, I encourage you to experiment with more knotwork on planar graphs.

Acknowledgements

I am very grateful to Christian Mercat for his instructive knotwork site [1]. I first learnt about this site when Christian was my officemate at the University of Melbourne in the early 2000s, and this article is heavily influenced by his approach.

Many thanks to Demetra Siountris and Jiayi Li for (independently) suggesting the use of Inkscape for the figures in this article and for their helpful tips and instructions.

References

[1] Christian Mercat, *Celtic Knotwork: the ultimate tutorial*, `http://www.entrelacs.net/-Celtic-Knotwork-The-ultimate-`, last accessed on 2022-03-11.

[2] Wikipedia, *Celtic knot*, `https://en.wikipedia.org/wiki/Celtic_knot`, last accessed on 2022-03-11.