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# Visualization of the third derivative of functions

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## 1 Introduction

In the first-year Calculus classes in high schools and colleges, visualization is an important tool for the understanding of abstract concepts such as derivatives. For example, the first derivative can be visualized as the slope of the tangent line. Typical textbooks in Calculus [1] discuss the second derivative in the context of concavity and the Second Derivative Test. They do not offer the direct visualization of the second derivative. Hardly any textbooks mention how to visualize the third derivative. In this article, we develop an interesting way to visualize the second and the third derivatives of single variable functions.

Visualization is both art and science. It targets to render an abstract concept into more basic entities and relationships through geometric representations. There may be multiple ways to visualize one concept. A good visualization should build on basic geometric elements, such as lengths, angles, and areas that can be drawn, at least conceptually, on the graph of the function. Derivatives involve limits. In this article, we will follow the traditional way of infinitesimal analysis to develop the visualizations.

## 2 Second Derivative

The first derivative of a function can be visualized as the slope of the tangent line. To visualize the second derivative, we need to analyze how the tangent line moves associated with the move of the independent variable of the function. In Figure 1, we draw the tangent line at two points.

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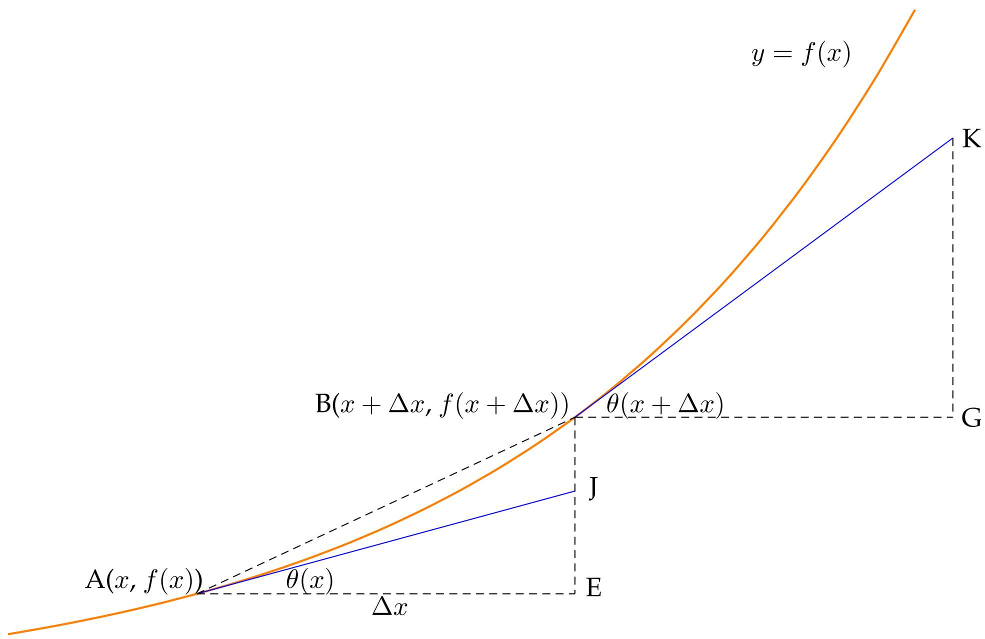



Figure 1

Let $A = (x, f(x))$ be a point on the graph of $y = f(x)$. We want to develop the visualization of the second and third derivatives at point $A$. For a small $\Delta x$, let point $B = (x + \Delta x, f(x + \Delta x))$ be the point on the graph of $y = f(x)$ where the independent variable moves from $x$ to $x + \Delta x$. Now let $E = (x + \Delta x, f(x))$ and we have $|\overline{AE}| = \Delta x$.

For the first derivative at point $A$, we draw the tangent line of $y = f(x)$ at point $A$ and let point $J$ be the intersection of that tangent line and line $BE$. From the definition of the derivative, $f'(x)$ is the slope of line $AJ$. Thus, $f'(x) = |\overline{JE}|/|\overline{AE}|$.

Now imagine that we move from point $A$ to point $B$ along the curve of $y = f(x)$. The tangent line $AJ$ will rotate and shift, and eventually move to line $BK$, which is the tangent line at point $B$. The derivative at point $B$ is $f'(x + \Delta x) = |\overline{KG}|/|\overline{BG}|$.

The second derivative of $f(x)$ at point $A$, $f''(x)$, is the limit of $(f'(x + \Delta x) - f'(x))/\Delta x$ as $\Delta x \to 0$. To visualize the second derivative $f''(x)$, we need to analyze two elements related to the rotation and shift of the tangent line: the tangential angle and the speed function.

## Tangential angle

The *tangential angle* $\theta(x)$ of function $f(x)$ at $x$ is defined as the measure (in radians) of the angle formed between the tangent line of the graph $y = f(x)$ at the point $(x, f(x))$ and the positive $x$-axis. We assume that $-\pi/2 < \theta(x) < \pi/2$. If the tangent line has a negative slope, then $\theta(x)$ is also negative.

From Figure 1, we see that line $AE$ is parallel to the $x$-axis and $\overrightarrow{AE}$ is pointing to the positive $x$ direction. Therefore, $\angle JAE$ is the angle between the tangent line $AJ$ and the positive $x$-axis; thus, $\theta(x) = \measuredangle JAE$. Similarly, we have $\theta(x + \Delta x) = \measuredangle KBG$.

## Speed function

As we will demonstrate next, another important element in the visualization of the second derivative $f''(x)$ is the speed function. Using the concepts from elementary differential geometry [2, p. 52], we can consider the graph of $y = f(x)$ as a two dimensional curve $\alpha(t) = (x(t), y(t))$ with the time variable $t = x$. That is, $(x(t), y(t)) = (t, f(t))$. Then, the velocity of the curve $\alpha(t)$ is $\alpha'(t) = (x'(t), y'(t)) = (1, f'(t))$. The *speed function*, $v(t)$, of the curve $\alpha(t)$, is defined as the magnitude of the velocity vector: $v(t) = \|\alpha'(t)\| = \sqrt{x'(t)^2 + y'(t)^2}$. Replacing $t$ with $x$, we can write the speed function of the graph of $y = f(x)$ in terms of $x$,

$$v(x) = \sqrt{1 + (f'(x))^2}\,. \tag{1}$$

In Figure 2 below, we imagine that as the independent variable moves from $x$ to $x+\Delta x$, the graph of $y = f(x)$ extends from point $A$ to point $B$. The speed function $v(x)$ is the rate of change of the length of the graph with respect to the change of the independent variable.

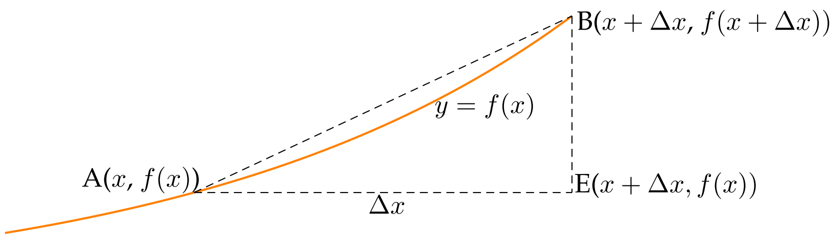



Figure 2

In fact, for a small $\Delta x$, the length of the graph between point $A$ and point $B$ can be approximated by $|\overline{AB}|$. Using $\triangle ABE$, we have

$$|\overline{AB}|^2 = |\overline{AE}|^2 + |\overline{BE}|^2 = \Delta x^2 + (f(x + \Delta x) - f(x))^2\,.$$

Therefore,

$$\lim_{\Delta x \to 0} \frac{|\overline{AB}|}{\Delta x} = \sqrt{1 + (f'(x))^2} = v(x)\,.$$

The definite integral of the speed function of a curve is called the arc length of the curve [2, p. 53]. For the graph of the function $y = f(x)$, $\int_a^b v(x)dx = \int_a^b \sqrt{1 + (f'(x))^2}dx$ is the arc length from point $(a, f(a))$ to point $(b, f(b))$. Thus, the speed function is the rate of the change of the arc length of the function.

## Visualization of the second derivative

From the definition of tangential angle $\theta(x)$, we have

$$\tan(\theta(x)) = f'(x)\,.$$

Taking the inverse function of the above, we get

$$\theta(x) = \arctan(f'(x))\,. \tag{2}$$

We differentiate both sides of Equation (2) with respect to variable $x$. Using the formula $(\arctan(u))' = u'/(1+u^2)$, we have

$$\theta'(x) = \frac{f''(x)}{1+(f'(x))^2}\,.$$

Rearranging the above equation and using Equation (1), we get

$$f''(x) = \theta'(x)(v(x))^2\,. \tag{3}$$

Equation (3) provides the basis for the visualization of the second derivative. As we move along the graph of the function $y = f(x)$, we observe a rotation of the tangent line and an extension of the arc length. Here, $\theta'(x)$ can be visualized as the angular velocity of the rotation of the tangent line, and $v(x)$ is the rate of the extension of the arc length (speed function).

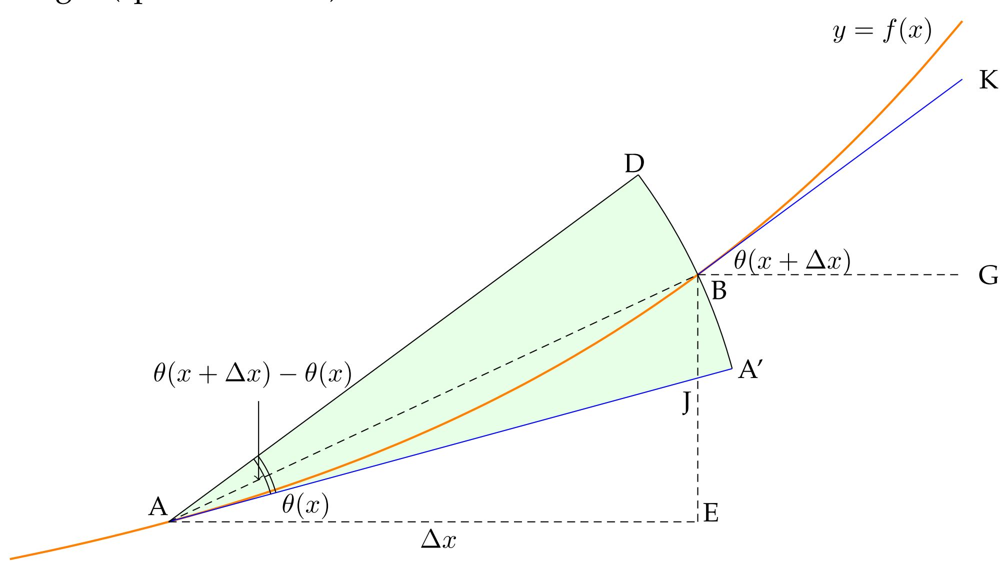



Figure 3

In Figure 3, we magnify the part of $\triangle ABE$ in Figure 1. We extend the tangent line $\overline{AJ}$ to point $A'$ with $|\overline{AA'}| = |\overline{AB}|$. Notice that line $BK$ is the tangent line at point $B$. In order to evaluate magnitude of rotation between the two tangent lines $AJ$ and $BK$, we draw a line passing through point $A$ and parallel to line $BK$ and extend the line to point $D$ such that $|\overline{AD}| = |\overline{AB}|$. Next, we draw an arc with center at point $A$ and the radius of $|\overline{AB}|$ which connects point $A'$, $B$, and $D$ by arc $A'D$.

Sector $AA'D$, shaded in green colour in Figure 3, is the key to the visualization of $f''(x)$. We know that the area of sector $AA'D$ is

$$\text{Area}(\text{Sector}(AA'D)) = \frac{1}{2}\measuredangle DAA'|\overline{AB}|^2 \,.$$

From the definition of tangential angle, we have $\measuredangle JAE = \theta(x)$ and $\measuredangle KBG = \theta(x+\Delta x)$. Notice that $AD \parallel BK$. Therefore,

$$\measuredangle DAA' = \measuredangle KBG - \measuredangle JAE = \theta(x+\Delta x) - \theta(x) \approx \theta'(x)\Delta x \,.$$

Furthermore, from the discussion of speed function, we have $|\overline{AB}| \approx v(x)\Delta x$. Therefore,

$$\text{Area}(\text{Sector}(AA'D)) = \frac{1}{2}\measuredangle DAA'|\overline{AB}|^2 \approx \frac{1}{2}\theta'(x)(v(x))^2))(\Delta x)^3 \,.$$

Together with Equation (3), we have

$$f''(x) = 2\lim_{\Delta x \to 0} \frac{\text{Area}(\text{Sector}(AA'D))}{(\Delta x)^3} \,.$$

For a small fixed $\Delta x$, the radius of the green shaded sector $AA'D$ is the visualization of $v(x)$ and the angle of the sector is the visualization of $\theta'(x)$. The area of sector $AA'D$ multiplying by 2 is the visualization of the second derivative $f''(x)$.

**Proposition 1.** *The visualization of the second derivative is 2 times the area of a sector whose angle is the angular velocity of the rotation of the tangent line and whose radius is the speed function.*

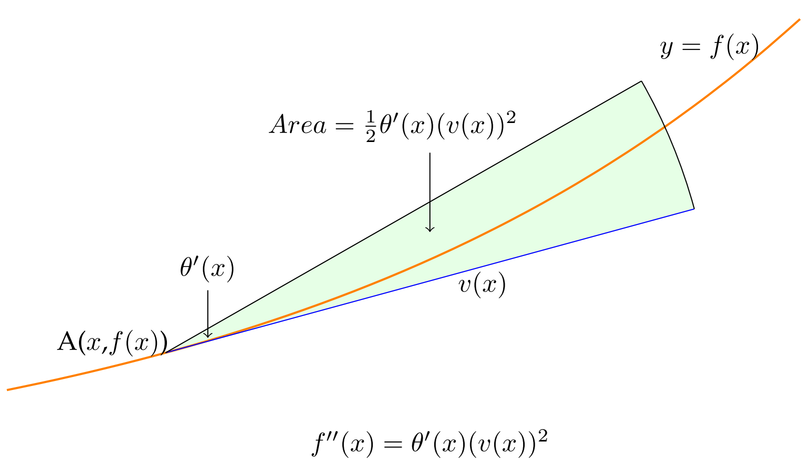
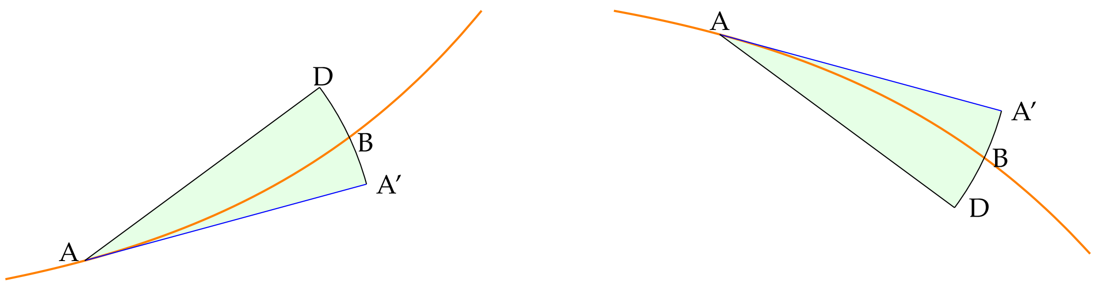



Figure 4

### The second derivative and the shape of the graph

The angular velocity $\theta'(x)$ can be positive or negative. As $x$ increases, if the tangent line rotates counter-clockwise, then $\theta'(x) > 0$. In Figure 3, this is the case that line $AD$ is above line $AA'$. If the tangent line rotates clockwise, then $\theta'(x) < 0$ (line $AD$ is below line $AA'$). Since $(v(x))^2 > 0$, the sign of $f''(x)$ is fully determined by the sign of $\theta'(x)$. The direction of the tangent line rotation is a good visualization of the concavity of the function.

**Proposition 2.** *If the tangent line rotates counter-clockwise when $x$ increases, then the function is concave upward. If the tangent line rotates clockwise when $x$ increases, then the function is concave downward.*




$\theta'(x) > 0$, Concave Upward

$\theta'(x) < 0$, Concave Downward

Figure 5

# 3 Third Derivative

With the preparations from the previous section, we can now develop the visualization of the third derivative. Equation (3) expresses the second derivative as the product of the angular velocity of the tangent line rotation and the square of the speed function. Differentiating both sides of Equation (3), we have

$$f'''(x) = \theta''(x)(v(x))^2 + 2\theta'(x)v(x)v'(x)\,. \tag{4}$$

To visualize the third derivative, we use the visualization of the second derivative from the previous section to study how the sector of the second derivative changes with respect to change of $x$.

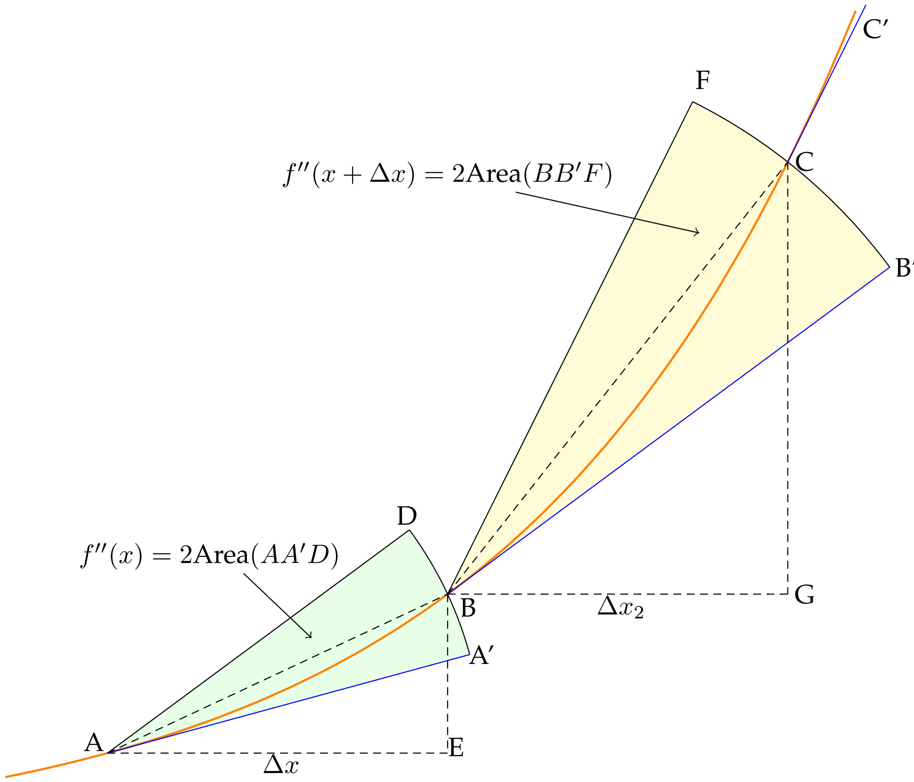



Figure 6

The third derivative is the rate of change of the second derivative,

$$f'''(x) = \lim_{\Delta x \to 0} \frac{f''(x+\Delta x) - f''(x)}{\Delta x}.$$

As shown in Figure 6, we know that the area of sector $AA'D$ multiplied by 2 is the visualization of $f''(x)$. To visualize $f''(x+\Delta x)$, we will construct the similar sector, sector $BB'F$, at point $B = (x+\Delta x, f(x+\Delta x))$. Here, we have $BF \parallel CC'$ where $CC'$ is the tangent line at point $C = (x+\Delta x+\Delta x_2, f(x+\Delta x+\Delta x_2))$ and $|\overline{BB'}| = |\overline{BF}| = |\overline{BC}|$. To visualize $f''(x+\Delta x) - f''(x)$, we need to subtract the area of sector $AA'D$ (green shade) from the area of sector $BB'F$ (yellow shade). To do so, we move sector $BB'F$ on top of sector $AA'D$ so that point $B$ coincides with point $A$ and the side $BB'$ overlies the side $AA'$.

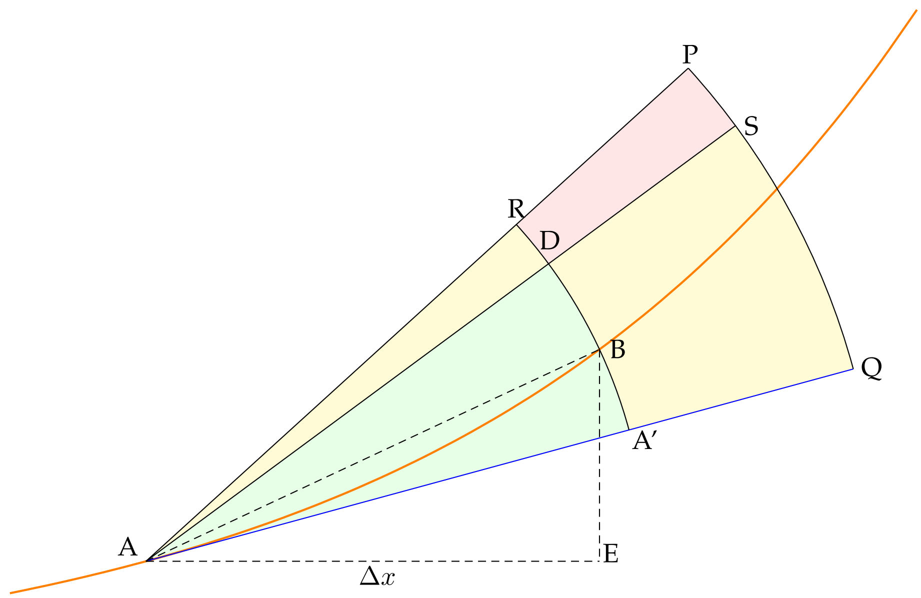



Figure 7

Sector $BB'F$ in Figure 6 is now moved to sector $AQP$ in Figure 7. In the case shown in Figure 7, sector $AQP$ is longer and wider than sector $AA'D$. In general, depending on the shape of the graph, Sector $AQP$ may be shorter and/or narrower than sector $AA'D$. Let point $R$ be the intersection of the extension of arc $A'D$ and line $AP$ and point $S$ be the intersection of arc $PQ$ and the extension of line $AD$. The difference between sector $AQP$ and sector $AA'D$ consists of three parts:

$$\text{Area}(AQP) - \text{Area}(AA'D) = \text{Area}(A'QSD) + \text{Area}(ADR) + \text{Area}(DSPR)\,.$$

The area Area$(DSPR)$ is of higher order in terms of $\Delta x$ and $\Delta x_2$ than Area$(A'QSD)$ and Area$(ADR)$, and is negligible as $\Delta x \to 0$ and $\Delta x_2 \to 0$.

## Rotational component

From Equation (4), we can see that $f'''(x)$ has two components. The first, $\theta''(x)(v(x))^2$, is associated with the rate of the change of the angular velocity (angular acceleration) of the rotation of the tangential line and can be visualized by sector $ADR$ in Figure 7. For sector $ADR$, its radius is $|\overline{AD}| = |\overline{AB}| \approx v(x)\Delta x$ and its angle,

$$\measuredangle DAR = \measuredangle RAA' - \measuredangle DAA' = \Delta\theta(x + \Delta x) - \Delta\theta(x)\,.$$

Here we can use second differential notation $\Delta^2$ and write

$$\Delta\theta(x + \Delta x) - \Delta\theta(x) = \Delta^2(\theta(x))\,.$$

Therefore,

$$\text{Area}(\text{Sector}(ADR)) = \frac{1}{2}|\overline{AD}|^2 \measuredangle DAR \approx \frac{1}{2}(v(x)\Delta x)^2 \Delta^2(\theta(x))\,.$$

Dividing both sides of the above the equation by $(\Delta x)^4$ and letting $\Delta x \to 0$, we have

$$\lim_{\Delta x \to 0} \frac{\text{Area}(\text{Sector}(ADR))}{(\Delta x)^4} = \frac{1}{2} \lim_{\Delta x \to 0} \frac{\Delta^2(\theta(x))}{(\Delta x)^2}(v(x))^2 = \frac{1}{2}\theta''(x)(v(x))^2 \,. \tag{5}$$

For a small fixed $\Delta x$ and $\Delta x_2$, the angle of sector $ADR$ is the visualization of $\theta''(x)$ and the radius of the sector is the visualization of $v(x)$. The area of sector $ADR$ multiplying by 2 is the visualization of the rotational component, $\theta''(x)(v(x))^2$, of the third derivative $f'''(x)$.

## Extension component

The second component of $f'''(x)$ in Equation (4), $2\theta'(x)v(x)v'(x)$, is associated with the rate of change of the speed function (the acceleration of the extension of the arc length) and can be visualized by disk sector $A'QSD$ in Figure 7. For $A'QSD$, its inner radius is $|\overline{AB}| \approx v(x)\Delta x$ and the outer radius is $|\overline{AQ}| \approx v(x+\Delta x)\Delta x$. The angle of the disk sector is the smaller one between $\measuredangle DAA' = \Delta\theta(x)$ and $\measuredangle PAQ = \Delta\theta(x+\Delta x)$. Thus, we have

$$\text{Area}(A'QSD) \approx \frac{1}{2}\Delta\theta(x)(v(x+\Delta x)+v(x))(v(x+\Delta x)-v(x))(\Delta x)^2 \,.$$

From this, we have

$$\lim_{\Delta x \to 0} \frac{\text{Area}(A'QSD)}{(\Delta x)^4} = \lim_{\Delta x \to 0} \frac{\Delta\theta(x)}{\Delta x} v(x) \frac{\Delta v(x)}{\Delta x} = \theta'(x)v(x)v'(x) \,. \tag{6}$$

For a small fixed $\Delta x$ and $\Delta x_2$, the angle of disk sector $A'QSD$ is the visualization of $\theta'(x)$. The inner radius of $A'QSD$ is the visualization of $v(x)$ and the difference between the outer radius and the inner radius of $A'QSD$ is the visualization of $v'(x)$. Thus, the area of disk sector $A'QSD$ multiplying by 2 is the visualization of the extension component, $2\theta'(x)v(x)v'(x)$, of the third derivative $f'''(x)$.

In summary, we know from Proposition 1 that the second derivative is 2 times the area of a sector whose angle is the angular velocity $\theta'(x)$ and whose radius is the speed function $v(x)$. As we move from $x$ to $x + \Delta x$, there are changes in both the angle $\theta'(x)$ and the radius $v(x)$ of the second derivative sector. The resulted change of the area of the sector consists of two components. The first one, $\theta''(x)(v(x))^2$, from the change of the angle of the sector, is associated with the angular acceleration of the rotation of the tangent line. The second component, $2\theta'(x)v(x)v'(x)$, from the change of the radius of the sector, is associated with the acceleration of the extension of the arc length.

**Proposition 3.** *The visualization of the third derivative is 2 times the sum of the areas of a sector and a disk sector (either area may have negative sign). For the sector, its angle is the angular acceleration of the rotation of the tangent line and its radius is the speed function. For the disk sector, its angle is the angular velocity of the rotation of the tangent line. One radius of the disk sector is the speed function and the other one is the speed function plus the rate of change of the speed function.*

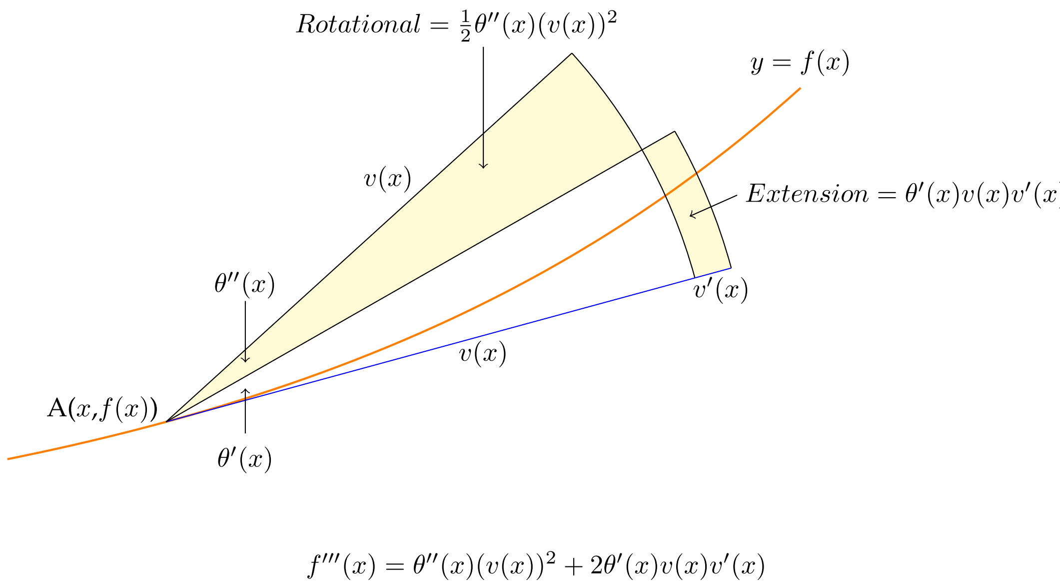



Figure 8

## Relationship with curvature

The concept of tangential angle $\theta(x)$ plays a key role in the visualization of the third derivative. It is the angle formed between the tangent line and the positive $x$-axis. The angle $\theta(x)$ is related to the concept of curvature $\kappa$ in elementary differential geometry [2, p. 68]. In fact, for a single variable function $y = f(x)$,

$$\kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{\frac{3}{2}}}\,.$$

Using speed function $v(x)$, when $f''(x) > 0$, we have

$$f''(x) = \kappa(x)(v(x))^3\,. \tag{7}$$

Differentiating both sides of the above equation gives us

$$f'''(x) = \kappa'(x)(v(x))^3 + 3\kappa(x)(v(x))^2 v'(x)\,. \tag{8}$$

It is difficult to make visualization of $f''(x)$ and $f'''(x)$ from Equation (7) and (8). The fundamental issue here is that curvature $\kappa$ is designed to measure how far a curve is deviated from the straight line in a coordinate-independent way. But for a single variable function $y = f(x)$, the definition of its derivatives are based on the existing $(x, y)$ coordinate system. That is, the derivatives of $f(x)$ are not coordinate-independent. This is why the tangential function $\theta(x)$, which measures the angle between the tangent line and the positive $x$-axis, is a better entity than $\kappa(x)$ to build the visualization of $f''(x)$ and $f'''(x)$.

## Relationship to the shape of the graph

As we discussed in Section 2, the equation $f''(x) = \theta'(x)(v(x))^2$ connects $f''(x)$ directly with shape of the graph of $y = f(x)$. $f''(x) > 0$ if and only if $\theta'(x) > 0$, which implies that the tangent line rotates counter-clockwise and the graph is concave upward. $f''(x) < 0$ if and only if $\theta'(x) < 0$, which implies that the tangent line rotates clockwise and the graph is concave downward.

Unfortunately, such direct relationship between the second derivative and the shape of the graph does not hold for the third derivative. From Equation (5), we can see that $f'''(x)$ is the sum of the rotational component (angular acceleration of the rotation of the tangent line), $\theta''(x)(v(x))^2$, and the extension component (the acceleration of the extension of the arc length), $2\theta'(x)v(x)v'(x)$. Because $f'''(x)$ is sum of the two parts, $f'''(x) > 0$ does not imply $\theta''(x) > 0$ nor $v'(x) > 0$. This is the reason that it is not easy to directly observe the relationship between $f'''(x)$ and the shape of the graph.

In the following tables, we present the data of the function $f(x) = x^3$ to highlight the issues associated with the relationship between the third derivative and the shape of the graph. In the table, $Rot = \theta''(x)(v(x))^2$ and $Ext = 2\theta'(x)v(x)v'(x)$. Though $f'''(x) = 6 > 0$ for all $x$, the angular velocity of the rotation of the tangent line stops increasing and the graph of $f(x) = x^3$ does not get curvier once $x$ passes a point around 0.4. As $x$ further increases, the angular velocity $\theta'(x)$ and angular acceleration $\theta''(x)$ gradually reduce to a negligible level and $f''(x)$ and $f'''(x)$ are influenced mainly by the speed function, $v(x)$, and the acceleration of the extension of the arc length, $v'(x)$. Thus, the third derivative alone may not offer much information about shape of the graph because the counter actions between the rotational component and the extension component.

$$f(x) = x^3$$

| $x$ | $f(x)$ | $f'(x)$ | $f''(x)$ | $f'''(x)$ | $v(x)$ | $v'(x)$ | $\theta'(x)$ | $\theta''(x)$ | $Rot$ | $Ext$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.10 | 0.00 | 0.03 | 0.60 | 6.00 | 1.00 | 0.01 | 0.59 | 5.97 | 5.97 | 0.02 |
| 0.20 | 0.00 | 0.12 | 1.20 | 6.00 | 1.00 | 0.14 | 1.18 | 5.57 | 5.65 | 0.34 |
| 0.30 | 0.02 | 0.27 | 1.80 | 6.00 | 1.03 | 0.46 | 1.67 | 4.07 | 4.36 | 1.63 |
| 0.40 | 0.06 | 0.48 | 2.40 | 6.00 | 1.10 | 1.03 | 1.95 | 1.22 | 1.50 | 4.49 |
| 0.50 | 0.12 | 0.75 | 3.00 | 6.00 | 1.25 | 1.80 | 1.92 | -1.69 | -2.64 | 8.64 |
| 0.60 | 0.21 | 1.08 | 3.60 | 6.00 | 1.47 | 2.64 | 1.66 | -3.20 | -6.93 | 12.92 |
| 0.70 | 0.34 | 1.47 | 4.20 | 6.00 | 1.77 | 3.47 | 1.32 | -3.30 | -10.41 | 16.40 |
| 0.80 | 0.51 | 1.92 | 4.80 | 6.00 | 2.16 | 4.25 | 1.02 | -2.75 | -12.88 | 18.87 |
| 0.90 | 0.72 | 2.43 | 5.40 | 6.00 | 2.62 | 4.99 | 0.78 | -2.11 | -14.53 | 20.52 |
| 1.00 | 1.00 | 3.00 | 6.00 | 6.00 | 3.16 | 5.69 | 0.60 | -1.56 | -15.60 | 21.60 |
| 2.00 | 8.00 | 12.00 | 12.00 | 6.00 | 12.04 | 11.95 | 0.08 | -0.13 | -17.84 | 23.83 |
| 5.00 | 125.00 | 75.00 | 30.00 | 6.00 | 75.00 | 29.99 | 0.00 | -0.01 | -18.00 | 23.99 |


# Acknowledgements

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# References


[1] R. Larson, R. Hostetler and B. Edwards, *Calculus of a single variable,* Sixth Edition, Houghton Mifflin, Boston, MA, 1998.

[2] B. O'Neill, *Elementary Differential Geometry*, Revised Second Edition, Elsevier, Burlington, MA, 2006.

# Appendix

## Geometric derivation of the second derivative formula

Equation (3) for the visualization of the second derivative, $f''(x) = \theta'(x)(v(x))^2$, is derived algebraically using formulae in calculus, especially the derivative of $\arctan(x)$. In this Appendix, we provide a direct geometric derivation of Equation (3).

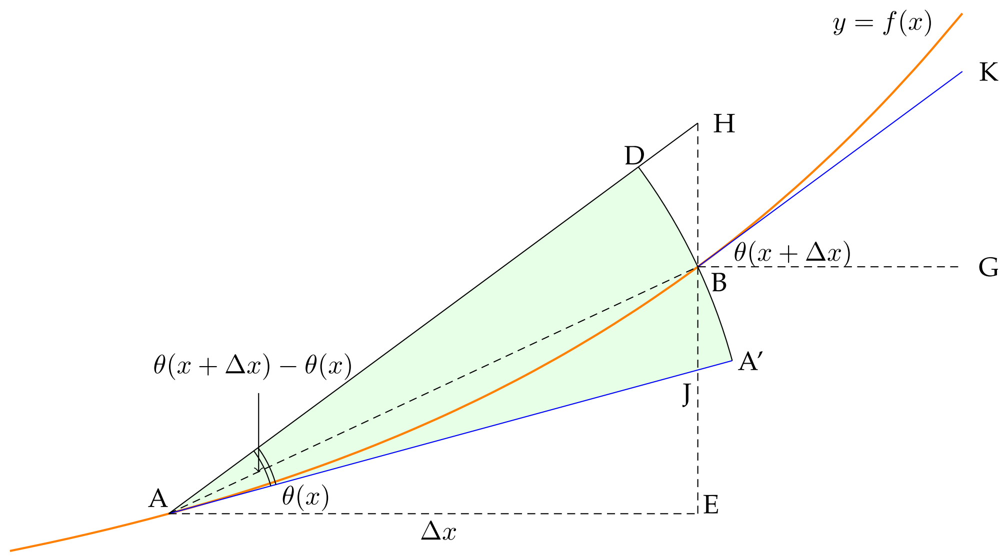


Figure 9

We modify Figure 3 slightly into Figure 9. Recall that $A = (x, f(x))$, $B = (x + \Delta x, f(x + \Delta x))$, $E = (x + \Delta x, f(x))$, line $AJ$ is the tangent line at point $A$, line $BK$ is the tangent line at point $B$. Line $AD$ is the line parallel to line $BK$. We extend line $AD$ to point $H$. Points $A'$, $B$ and $D$ are connected by arc $A'BD$ with $|\overline{AD}| = |\overline{AB}| = |\overline{AA'}|$.

From the definition of tangential angle, we have $\theta(x) = \measuredangle JAE$. Furthermore, the slope of line $AJ$ is the first derivative $f'(x)$. Thus,

$$f'(x) = |\overline{JE}|/|\overline{AE}| = |\overline{JE}|/\Delta x\,.$$

Similarly, since $AH \| BK$, $\theta(x + \Delta x) = \measuredangle KBG = \measuredangle HAE$ and the slope of line $AH$ is $f'(x + \Delta x)$,

$$f'(x + \Delta x) = |\overline{HE}|/|\overline{AE}| = |\overline{HE}|/\Delta x\,.$$

Therefore,

$$|\overline{HJ}| = |\overline{HE}| - |\overline{JE}| = (f'(x + \Delta x) - f'(x))\Delta x\,.$$

$\overline{HJ}$ and $\overline{AE}$ form the base and the height of $\triangle AHJ$. Thus,

$$\text{Area}(\triangle AHJ) = \frac{1}{2}|\overline{HJ}||\overline{AE}| = \frac{1}{2}(f'(x + \Delta x) - f'(x))(\Delta x)^2\,.$$

From the definition of the second derivative,

$$f''(x) = \lim_{\Delta x \to 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\text{Area}(\triangle AHJ)}{(\Delta x)^3} \,. \tag{9}$$

From Figure 9, we can see that $\angle HAJ$ in $\triangle AHJ$ is the difference between two tangential angles:

$$\measuredangle HAJ = \measuredangle HAE - \measuredangle JAE = \theta(x + \Delta x) - \theta(x) = \Delta\theta(x) \,. \tag{10}$$

Using the formula of the area of a triangle, $Area = \frac{1}{2}ab\sin(C)$, we have

$$\text{Area}(\triangle AHJ) = \frac{1}{2}|\overline{AH}||\overline{AJ}|\sin(\measuredangle HAJ) \,. \tag{11}$$

Using the notion of infinitesimal approach, from Figure 9, we observe that as $\Delta x \to 0$, both $|\overline{AH}| \approx |\overline{AB}|$ and $|\overline{AJ}| \approx |\overline{AB}|$. Furthermore, using the formula in calculus, $\lim_{\theta \to 0} \sin(\theta)/\theta = 1$, we have $\sin(\measuredangle HAJ) \approx \measuredangle HAJ$ as $\Delta x \to 0$.
Now substituting $|\overline{AB}|$ for $|\overline{AH}|$ and $|\overline{AJ}|$ and $\measuredangle HAJ$ for $\sin(\measuredangle HAJ)$ in Equation (11), we have Area$(\triangle AHJ) = \frac{1}{2}|\overline{AH}||\overline{AJ}|\sin(\measuredangle HAJ) \approx \frac{1}{2}|\overline{AB}|^2 \measuredangle HAJ$ as $\Delta x \to 0$. Notice that $\frac{1}{2}|\overline{AB}|^2 \measuredangle HAJ$ is exactly the area of sector $AA'D$. Together with Equation (9),

$$f''(x) = \lim_{\Delta x \to 0} \frac{2\text{Area}(\text{Sector}(AA'D))}{(\Delta x)^3} = \lim_{\Delta x \to 0} \frac{|\overline{AB}|^2}{(\Delta x)^2} \frac{\measuredangle HAJ}{\Delta x} \,. \tag{12}$$

From the definition of speed function, we have

$$\lim_{\Delta x \to 0} \frac{|\overline{AB}|}{\Delta x} = v(x) \,.$$

By Equation (10), we get

$$\lim_{\Delta x \to 0} \frac{\measuredangle HAJ}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta\theta(x)}{\Delta x} = \theta'(x) \,.$$

Together with Equation (12), we thereby prove that

$$f''(x) = \theta'(x)(v(x))^2 \,.$$