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A generalisation of the Arithmetic-Logarithmic-Geometric Mean Inequality

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1 Introduction

It was a usual boring day when I was searching for something new, and the discontinued publication *Mathematical Spectrum* came to mind. In the back issues of the magazine[2], I found a letter to the editor written by Spiros P. Andriopoulos [1]. He referred to an inequality that was published in the *Octogon Mathematical Magazine* [3] that proves, for all positive real numbers a and b, the Arithmetic-Logarithmic-Geometric Mean Inequality:

$$\sqrt{ab} < \frac{a-b}{\ln a - \ln b} < \frac{a+b}{2}\,.$$

In this article, we use the Hermite-Hadamard Inequality [6, 9] to prove the following inequality:

Theorem 1. *Let n be a non-negative integer. If $x > y > 0$, then*

$$\begin{aligned}\sqrt{xy}(\ln\sqrt{xy})^{n-1}(\ln\sqrt{xy}+n) &< \frac{x(\ln x)^n - y(\ln y)^n}{\ln x - \ln y}\\ &< \frac{x(\ln x)^{n-1}(\ln x + n) + y(\ln y)^{n-1}(\ln y + n)}{2}\,.\end{aligned}$$

It is the first time that I have come across such a powerful inequality that is applicable for convex functions.

By setting $n = 0$ in the expressions above, we have

$$\sqrt{xy}(\ln\sqrt{xy})^{-1}(\ln\sqrt{xy}) < \frac{x-y}{\ln x - \ln y} < \frac{x+y}{2}\,.$$

Therefore, we obtain the following logarithm extension to the Arithmetic Mean and Geometric Mean Inequality (AM-GM Inequality) [10]:

Corollary 2.

$$\sqrt{xy} < \frac{x-y}{\ln x - \ln y} < \frac{x+y}{2}\,.$$

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[2]These back issues are now kindly provided by the *Applied Probability Trust* on their website https://appliedprobability.org/.

2 The Hermite-Hadamard Inequality

A function $f(x)$ is *convex* in an interval $[a, b]$ if the second derivative $f''(x)$ is non-negative for all $x \in (a, b)$. Convex functions satisfy the Hermite-Hadamard Inequality:

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}\,. \tag{1}$$

This inequality is named after the famous mathematicians Charles Hermite and Jacques Hadamard, whose photos [8, 11] are shown here:

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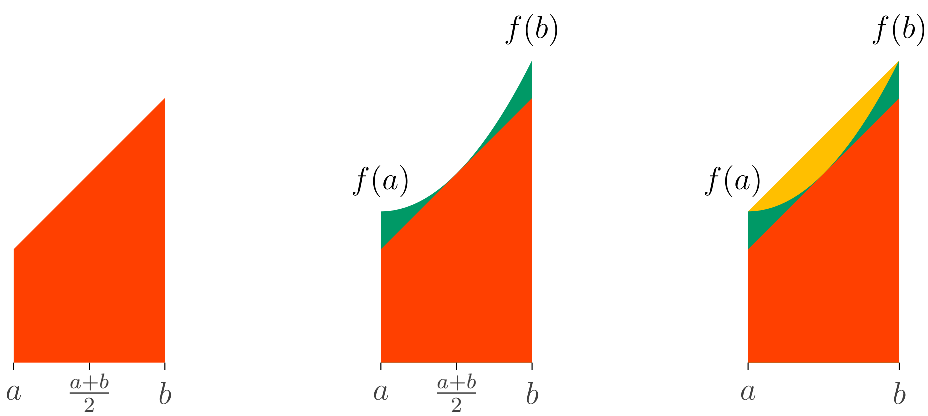
Charles Hermite (1822-1901) Jacques Hadamard (1865-1963)

We provide a simple graphical proof of the Hermite-Hadamard Inequality.

In particular, consider the following three shapes:



The middle shape is the area beneath the curve of the function $f(x)$ for $x \in [a, b]$. The first shape shows the area beneath the tangent to the curve of $f(x)$ in the point $x = (a + b)/2$. The last shape shows the area beneath the line from point $\big(a, f(a)\big)$ to point $\big(b, f(b)\big)$ Expressed mathematically, these areas are, respectively,

$$(b-a)f\Big(\frac{a+b}{2}\Big)\,, \qquad \int_a^b f(x)dx\,, \qquad (b-a)\frac{f(a)+f(b)}{2}\,.$$

Since $f(x)$ is convex, the three areas are non-decreasing in size; so

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}\,.$$

This is the Hermite-Hadamard Inequality, which we have hereby proved.

Next, we introduce two examples of Hermite-Hadamard Inequality, the first of which is the following lemma, due to Spiros P. Andriopoulos [2].

Lemma 3. *For all* $x \in (0, \frac{\pi}{2})$,

$$\frac{e^{\sin x} - e^{\tan x}}{\sin x - \tan x} > e^x\,.$$

Proof. Let $f(x) = e^x$. By the Hermite-Hadamard Inequality,

$$\frac{e^b - e^a}{b-a} = \frac{\int_a^b e^x dx}{b-a} > e^{\frac{a+b}{2}}\,.$$

Therefore, for all $x \in (0, \frac{\pi}{2})$,

$$\frac{e^{\sin x} - e^{\tan x}}{\sin x - \tan x} > e^{(\frac{\sin x + \tan x}{2})}\,. \tag{2}$$

According to [5], $\sin x + \tan x > 2x$ for all $x \in (0, \frac{\pi}{2})$, so, by applying this inequality to (2), we find that

$$\frac{e^{\sin x} - e^{\tan x}}{\sin x - \tan x} > e^x\,.$$

□

The second example is an inequality proposed by Dorin Marghidanu [4].

Lemma 4. *If* $b > a$, *then*

$$\frac{a+b}{2} < \ln\frac{e^b - e^a}{b-a} < \ln\frac{e^b + e^a}{2} < \frac{e^b + e^a - 2}{2}\,.$$

Proof. Let $f(x) = e^x$. By the Hermite-Hadamard Inequality,

$$e^{\frac{a+b}{2}} \le \frac{\int_a^b e^x dx}{b-a} = \frac{e^b - e^a}{b-a} \le \frac{e^b + e^a}{2}\,,$$

so

$$\frac{a+b}{2} < \ln\frac{e^b - e^a}{b-a} < \ln\frac{e^b + e^a}{2}\,.$$

According to [7], $\ln x < x - 1$ for all $x > 0$, so

$$\ln\frac{e^b + e^a}{2} < \frac{e^b + e^a - 2}{2}\,.$$

Therefore,

$$\frac{a+b}{2} < \ln\frac{e^b - e^a}{b-a} < \frac{e^b + e^a - 2}{2}\,.$$

□

3 Proof of Theorem 1

Let k be a positive integer, suppose that $b > a > 0$, and define $f(x) = x^k e^x$ for all $x \in [a, b]$. By the Hermite-Hadamard Inequality and partial integration,

$$\left(\frac{a+b}{2}\right)^k e^{\frac{a+b}{2}} \le \frac{1}{b-a}\int_a^b x^k e^k dx = \frac{b^k e^b - a^k e^a}{b-a} - \frac{k}{b-a}\int_a^b x^{k-1}e^x dx\,,$$

so

$$\left(\frac{a+b}{2}\right)^k e^{\frac{a+b}{2}} + \frac{k}{b-a}\int_a^b x^{k-1}e^x dx \le \frac{b^k e^b - a^k e^a}{b-a}\,.$$

Now define $f(x) = x^{k-1}e^x$ for all $x \in [a, b]$. By the Hermite-Hadamard Inequality,

$$\left(\frac{a+b}{2}\right)^{k-1} e^{\frac{a+b}{2}} \le \frac{1}{b-a}\int_a^b x^{k-1}e^{k-1} dx\,.$$

so

$$\left(\frac{a+b}{2}\right)^k e^{\frac{a+b}{2}} + k\left(\frac{a+b}{2}\right)^{k-1} e^{\frac{a+b}{2}} \le \left(\frac{a+b}{2}\right)^k e^{\frac{a+b}{2}} + \frac{k}{b-a}\int_a^b x^{k-1}e^x dx \le \frac{b^k e^b - a^k e^a}{b-a}\,.$$

Now substitute $e^b = x$, $e^a = y$ and $k = n$:

$$\begin{aligned}\sqrt{xy}(\ln\sqrt{xy})^{n-1}(\log\sqrt{xy}+n) &= \sqrt{xy}\big(\ln\sqrt{xy}\big)^n + n\sqrt{xy}(\ln\sqrt{xy})^{n-1}\\ &= \left(\frac{\ln y + \ln x}{2}\right)^n e^{\left(\frac{\ln y+\ln x}{2}\right)} + n\left(\frac{\ln y + \ln x}{2}\right)^{n-1} e^{\left(\frac{\ln y+\ln x}{2}\right)}\\ &\le \frac{x\ln^n x - y\ln^n y}{\ln x - \ln y}\,.\end{aligned}$$

This proves one inequality of Theorem 1. The other inequality is proved similarly. □

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