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Non-negative numbers and infinitely nested square roots

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1 Introduction

The study of square roots has always been fascinating to me. I've always loved solving mathematics problems that heavily involved square root manipulation; it's pretty elegant when a complicated square root expression, which is typically not computable to exact precision, can be broken down into a result that most of us can understand. My article shows you the beauty of square roots, and how non-negative numbers can be expressed as infinitely nested square roots. A decent understanding of sequences, mathematical proofs and, of course, square roots is an important prerequisite before reading my paper.

2 What I seek to prove

Let's get a little more technical. In my paper, I prove that any number greater than 1 can be expressed as an infinitely nested square root of the form $\sqrt{k+\sqrt{k+\sqrt{k+\cdots}}}$ for some non-negative constant k. To put it more formally,

"For any real number $N > 1$,

$$N = \sqrt{k+\sqrt{k+\sqrt{k+\cdots}}} \quad \text{for some non-negative constant } k."$$

2.1 The necessary proof of convergence

Before we make sense of the expression, we need to show that for any non-negative constant k, $\sqrt{k+\sqrt{k+\sqrt{k+\cdots}}}$ is convergent. The proof of convergence would remove any ambiguity related to infinity.

2.2 Addressing the cases $N = 0$ and $N = 1$

After our proof of convergence, I will be addressing the cases where $N = 0$ and $N = 1$. These cases need to be considered separately, for reasons you will discover as you delve deeper into the paper.

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2.3 The proof

Once we get that out of the way, we'll go into the exact details of how any real number greater than 1 can be expressed as an infinitely nested square root of the form $\sqrt{k+\sqrt{k+\sqrt{k+\cdots}}}$ for some non-negative constant k.

3 Convergence

Let's begin by proving that for any non-negative constant k, $\sqrt{k+\sqrt{k+\sqrt{k+\cdots}}}$ is convergent. To do this, let's use a recurrence relation to model this expression:

$$x_1 = \sqrt{k} \qquad \text{and} \qquad x_{n+1} = \sqrt{k + x_n}\,.$$

We now need to prove the convergence of $\{x_n\}$ as n approaches infinity. This will be done in a two-step process - firstly, we'll show that $\{x_n\}$ is non-decreasing and, next, we'll show that $\{x_n\}$ is bounded. By the Monotone Convergence Theorem, $\{x_n\}$ is convergent if these two conditions are proven.

3.1 $\{x_n\}$ is non-decreasing

We will use induction to prove that $\{x_n\}$ is non-decreasing, by proving that $x_{n+1} \geq x_n$ for each integer $n \geq 1$. First, note that since $\sqrt{k} \geq 0$,

$$x_2 = \sqrt{k+\sqrt{k}} \geq \sqrt{k} = x_1\,.$$

Next, assume that $x_{n+1} \geq x_n$ for some integer $n \geq 1$. Then

$$x_{n+2} = \sqrt{k + x_{n+1}} \geq \sqrt{k + x_n} = x_{n+1}\,.$$

By induction, it follows that $x_{n+1} \geq x_n$ for each integer $n \geq 1$. Therefore, $\{x_n\}$ is non-decreasing.

3.2 $\{x_n\}$ has an upper bound

Define $M = \frac{1}{2} + \sqrt{k + \frac{1}{4}}$. We will use induction to prove that $\{x_n\}$ has an upper bound by proving that $x_n \leq M$ for each integer $n \geq 1$. First, note that since $k \geq 0$, we know that $\sqrt{k+\frac{1}{4}} > \sqrt{k}$, and

$$x_1 = \sqrt{k} \leq \frac{1}{2} + \sqrt{k + \frac{1}{4}} = M\,.$$

Next, assume that $x_n \leq M$ for some integer $n \geq 1$. Since $M = \frac{1}{2} + \sqrt{k+\frac{1}{4}}$,

$$M^2 = \left(\frac{1}{2} + \sqrt{k+\frac{1}{4}}\right)^2 = k + \frac{1}{2} + \sqrt{k + \frac{1}{4}} = k + M\,.$$

It follows that $M = \sqrt{k + M}$ since $M \geq 0$. Then, by the assumption $x_n \leq M$,

$$x_{n+1} = \sqrt{k + x_n} \leq \sqrt{k + M} = M\,.$$

By induction, $x_n \leq M$ for all integers $n \geq 1$; that is, $\{x_n\}$ is bounded above by M.

We have now completed our two-step proof and conclude that $\{x_n\}$ is convergent.

4 Addressing two unique cases

We will now address the two unique cases where $N = 0$ and $N = 1$. For $N = 0$,

$$0 = \sqrt{0 + \sqrt{0 + \sqrt{0 + \cdots}}}.$$

The case in which $N = 1$ is more interesting. Assume that we have

$$1 = \sqrt{k + \sqrt{k + \sqrt{k + \cdots}}}$$

for some non-negative number k. Then square both sides:

$$1 = k + \sqrt{k + \sqrt{k + \sqrt{k + \cdots}}} = k + 1\,.$$

We obtain $k = 0$ as our solution. However, this leads to the contradiction

$$1 = \sqrt{0 + \sqrt{0 + \sqrt{0 + \cdots}}} = 0\,,$$

so our assumption is false. Therefore, 1 cannot be written as $\sqrt{k + \sqrt{k + \sqrt{k + \cdots}}}$ for any non-negative integer k.

5 The proof

We will now show that any real number $N > 1$ can be expressed as an infinitely nested root of the form $\sqrt{k + \sqrt{k + \sqrt{k + \cdots}}}$ for some non-negative constant k.

In particular, define

$$k = N(N-1) = N^2 - N\,. \tag{1}$$

Then $N^2 = k + N$, so

$$N = \sqrt{k+N} = \sqrt{k + \sqrt{k+N}} = \cdots = \sqrt{k + \sqrt{k + \sqrt{k + \cdots}}}\,,$$

which is what we wanted to prove. Also note that for non-negative k, we have $N^2 - N \geq 0$, and as such, $N \leq 0$ or $N \geq 1$. As N is non-negative, we can take the solution of this inequality to be $N = 0$ or $N \geq 1$. However, following the unique cases addressed in Section 4, we see that the case $N = 1$ leads to a contradiction, so we can conclude that any real number $N > 1$ can be expressed in the stated form.

6 Application

Suppose we want to express the number $N = 5$ as an infinitely nested square root of the form $\sqrt{k + \sqrt{k + \sqrt{k + \cdots}}}$ for some non-negative constant k. We can now use Equation (1) from Section 5 above to define

$$k = N(N-1) = 5 \times (5-1) = 20\,.$$

Then

$$5 = \sqrt{20 + \sqrt{20 + \sqrt{20 + \cdots}}}\,.$$

A quick punch of the calculator would suggest that the above equation is indeed valid.

7 The golden ratio

Now, you may be wondering what the golden ratio ϕ has to do with this. Surprisingly enough, the nested square roots discussed in this paper yield a beautiful expression for the golden ratio. As you might know, the golden ratio is obtained from the following quadratic equation:

$$\phi^2 = \phi + 1$$

or, equivalently, $1 = \phi(\phi - 1)$. This is similar in form to Equation (1). Indeed, by setting $N = \phi$ and $k = 1$, we have:

$$\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}\,.$$

8 Conclusion

I believe that infinitely nested square roots are truly beautiful. We've seen clearly from this example that any non-negative number, excluding numbers greater than 0 but lesser than or equal to 1, can be expressed as an infinitely nested square root with form $\sqrt{k+\sqrt{k+\sqrt{k+\cdots}}}$ for some non-negative constant k. My work showcases just one of many kinds of infinitely nested radicals, and there is a great deal of exploration that can be done beyond this; see for instance [1]. Whether it's alternating signs within the radicals, or more complex radicals nested within, or roots of different degrees, the sky is the limit.

References

[1] R. Schneider, Fibonacci numbers and the golden ratio, *Parabola* **52 (3)** (2016).