Parabola Volume 58, Issue 2 (2022)

Solutions 1671–1680

Q1671 As in Problem 1643 and Problem 1663, a Geezer number is a k–digit positive integer $d_0d_1\cdots d_{k-1}$ in base 10 which consists of d_0 zeros, d_1 ones and so on. We have already proved that, in a k–digit Geezer number, the sum of the digits is k; and the digits $d_3, d_4, \ldots, d_{k-1}$ include at most one 1, with all the rest of these digits being 0. Find all Geezer numbers.

SOLUTION Let n be a k–digit Geezer number. Then the sum of the digits in n is also k. Let the first digit of n be f. Then n contains exactly f zeros and at least one f. There are another $k-f-1$ non–zero digits with sum $k-f$; the only way that this is possible is if one of these digits is 2 and all the rest are 1s. So the digits of n are 0 occurring f times, 2 occurring once, possibly some 1s, and a digit f which **may** be another 1 or 2.

- Suppose that $f \geq 3$. Then $d_f \geq 1$; from previous results, this means that $d_f = 1$ and all other digits $d_3, d_4, \ldots$ are 0. Just one of the digits is 2, so $d_2 = 1$. The digits d_2 and d_f are 1, so $d_1 = 2$. The digits are 0 occurring f times, 1 occurring twice, 2 occurring once and f occurring once. The sum of digits is $f + 4$, so $f \leq 6$. The non–zero digits preceding d_f are d_0, d_1, d_2, so there are $f - 3$ zeros before d_f, and 3 zeros afterwards. This gives the Geezer numbers
$$6210001000\,,\quad 521001000\,,\quad 42101000\,,\quad 3211000\,.$$
- Suppose that $f = 2$. Then the digits are 0 twice, 2 twice and possibly some 1s. So n is $2\,d_1 2\cdots$ with $d_1 = 0$ or 1, and we have Geezer numbers
$$2020\,,\quad 21200\,.$$
- Suppose that $f = 1$. Then we have 0 once, 2 once and some 1s. The only digit which could be 2 is d_1, so our last Geezer number is
$$1210\,.$$

Q1672 A bag contains $2n$ balls, two each of n different colours. They are to be drawn from the bag in a random order and placed in a row. Prove *without calculation* that the probability of obtaining a row consisting of pairs of the same colour is the same as the probability of obtaining a row in which the second half is the same as the first.

SOLUTION Imagine that two people each have a row of empty places numbered $1, 2, \ldots, 2n-1, 2n$ in which they will write the colours of the balls. A third person draws balls from the bag one by one and announces the colours as they are drawn. The first person writes colours into places in the order $1, 2, 3, 4, \ldots, 2n-1, 2n$; the second does so in the order $1, n+1, 2, n+2, \ldots, n, 2n$. Then the first person will obtain a row of pairs if and only if the second obtains a row of two identical halves. Therefore, the probabilities of the two outcomes are the same.

Q1673 The article by Jonathan Hoseana and Handi Koswara in *Parabola* Volume 57, Issue 2 may help with the following problem.

(a) Find a constant α such that the function

$$\alpha x - \sum_{i=1}^{n} 2^i \sqrt{x^2 + \frac{x}{4^i}} \qquad (*)$$

has a limit as $x \to \infty$.

(b) If α is the value in (a) and L_n is the corresponding limit (in terms of n), then find the limit of L_n as $n \to \infty$.

SOLUTION Using the notation of the paper mentioned in the question, we have $A = \alpha^2$, $B = 0$ and

$$a_i = (2^i)^2 = 4^i\,, \quad b_i = \frac{(2^i)^2}{4^i} = 1$$

for $i = 1, 2, \ldots, n$. The limit of $(*)$ exists if

$$\sqrt{A} = \sum_{i=1}^{n} \sqrt{a_i}\,,$$

that is, if

$$\alpha = \sum_{i=1}^{n} 2^i = 2^{n+1} - 2\,.$$

In this case, the value of the limit is

$$L_n = \frac{B}{2\sqrt{A}} - \sum_{i=1}^{n} \frac{b_i}{2\sqrt{a_i}} = -\sum_{i=1}^{n} \frac{1}{2^{i+1}}\,,$$

and the limit of L_n as $n \to \infty$ is given by an infinite geometric progression,

$$\lim_{n\to\infty} L_n = -\sum_{i=1}^{\infty} \frac{1}{2^{i+1}} = -\frac{1}{2}\,.$$

A solution was submitted by Hyunbin Yoo, South Korea.

Q1674 A sequence of numbers $x_1, x_2, x_3, \ldots$ is generated as follows. We begin with $x_1 = 1$; then we take the cosine and sine of x_1; then the cosine and sine of x_2; and so on:

$$\begin{aligned}
x_1 &= 1\\
x_2 &= \cos(x_1) = \cos(1)\\
x_3 &= \sin(x_1) = \sin(1)\\
x_4 &= \cos(x_2) = \cos(\cos(1))\\
x_5 &= \sin(x_2) = \sin(\cos(1))
\end{aligned}$$

$$x_6 = \cos(x_3) = \cos(\sin(1))$$
$$x_7 = \sin(x_3) = \sin(\sin(1))$$
$$x_8 = \cos(x_4) = \cos(\cos(\cos(1)))$$

and so on. Find the smallest $n > 1$ such that $x_n > 0.99$.

SOLUTION A general formula for the sequence is $x_1 = 1$ and

$$x_{2k} = \cos(x_k)\,, \quad x_{2k+1} = \sin(x_k)$$

for all positive integers k. We shall consider the numbers x_n in sets of $1, 2, 4, 8, \ldots$ by defining

$$S_j = \{n \in \mathbb{Z} \mid 2^j \le n < 2^{j+1}\} \quad \text{and} \quad X_j = \{x_n \mid n \in S_j\}\,.$$

First we shall show that, for each $j \ge 2$, the largest element in X_j is

$$x_{2^j+2^{j-1}-2} = \cos(\sin(\sin\cdots(\sin(\cos 1))\cdots))\,,$$

where there are $j-2$ sine terms in the left–hand side; and the smallest is

$$x_{2^j+2^{j-1}-1} = \sin(\sin(\sin\cdots(\sin(\cos 1))\cdots))\,,$$

where there are $j-1$ sine terms in the left–hand side. To confirm that this is true for $j = 2$, we simply calculate the four terms in X_2: we find

$$x_4 = 0.85\,, \quad x_5 = 0.51\,, \quad x_6 = 0.66\,, \quad x_7 = 0.74\,,$$

and clearly the largest is x_4 and the smallest is x_5. Now we continue by induction, showing that if the claimed facts are true for some specific j, then they are also true for $j+1$. It is clear that all values of x_n lie between 0 and 1. Therefore, when we form X_{j+1} by calculating the cosine and sine of all elements of X_j, the largest element will be either the cosine of the smallest element in X_j or the sine of the largest element in X_j. By assumption, these possibilities are

$$x_{2^{j+1}+2^j-2} = \cos(x_{2^j+2^{j-1}-1}) = \cos(\sin(\sin\cdots(\sin(\cos 1))\cdots))$$

with $j-1$ sine terms, and

$$x_{2^{j+1}+2^j-3} = \sin(x_{2^j+2^{j-1}-2}) = \sin(\cos(\sin\cdots(\sin(\cos 1))\cdots))$$

with $j-2$ sine terms, not counting the first one. To determine which of these is the larger, note that

$$x_{2^{j+1}+2^j-2} > \cos(\cos 1) > 0.85 \quad \text{and} \quad x_{2^{j+1}+2^j-3} < \sin 1 < 0.85\,;$$

so the largest element of X_{j+1} is $x_{2^{j+1}+2^j-2}$, as claimed. The argument for the smallest element is very similar. The choice lies between

$$x_{2^{j+1}+2^j-4} = \cos(x_{2^j+2^{j-1}-2}) > \cos 1 > 0.54$$

and

$$x_{2^{j+1}+2^j-1} = \sin(x_{2^j+2^{j-1}-1}) < \sin(\cos 1) < 0.52\,;$$

clearly the latter is the smaller, and is therefore the smallest element of X_{j+1}. This completes the proof.

Next, we note that the maximum element of X_j increases as j increases. This is because if we write

$$\theta = \sin(\sin\cdots(\sin(\cos 1))\cdots)$$

with $j-2$ sine terms, then $\sin\theta < \theta$ and so

$$x_{2^{j+1}+2^j-2} = \cos(\sin\theta) > \cos\theta = x_{2^j+2^{j-1}-2}\,.$$

It follows that the maximum element in any of the sets $X_0, X_1, X_2, \ldots, X_j$ is $x_{2^j+2^{j-1}-2}$. How long does it take to make this maximum greater than 0.99? We need

$$\sin(\sin\cdots(\sin(\cos 1))\cdots) < \arccos(0.99) = 0.141539\cdots\,,$$

where there are $j-2$ sine terms on the left–hand side. So, enter 1 into a calculator, hit the "cos" button once and then the "sin" button repeatedly, keeping count of how many times you hit it, until the result satisfies this inequality. If you have a programmable calculator or equivalent software, you will be able to automate this. Either way, you will find that 137 sine terms is not enough but 138 is. In other words, the greatest element in any of the sets $X_0, X_1, X_2, \ldots, X_{139}$ is

$$x_{2^{139}+2^{138}-2} = \cos(\sin(\sin\cdots(\sin(\cos 1))\cdots)) = 0.989943\cdots\,,$$

which is still less than 0.99, and the greatest element in X_{140} is

$$x_{2^{140}+2^{139}-2} = 0.990010\cdots\,,$$

which is greater than 0.99. This looks like our answer, but we should be careful: there are many terms in X_{140} before we get to this one ($2^{139}-2$ of them to be precise), and it is conceivable that one of them, although not the greatest element in X_{140}, might already be greater than 0.99. To resolve this question, we consider the *second* largest and *second* smallest elements in X_j. We shall show that if $j \geq 3$, then the second largest element of X_j is

$$x_{2^j+2^{j-1}+2^{j-2}-2} = \cos(\sin(\sin\cdots(\sin(\cos(\sin 1)))\cdots)))$$

with $j-3$ sine terms in the middle, not counting the last one; and the second smallest is

$$x_{2^j+2^{j-1}+2^{j-2}-1} = \sin(\sin(\sin\cdots(\sin(\cos(\sin 1)))\cdots))) \qquad (*)$$

starting with $j-2$ sine terms. The proof is much the same as we did before, though, not surprisingly, we have to be a little more careful with the details. First, we check the statement by direct calculation for $j = 3, 4, 5$: this is a bit of work but is basically routine. Now suppose that the claim is true for some specific $j \geq 5$. The choice for the second largest element of X_{j+1} lies between the cosine of the second smallest element in X_j (because we know that the cosine of the smallest gives the largest element in X_{j+1}, not the second largest) and the sine of the largest. Noting that the number of sine terms in $(*)$ is at least 3, the element that we are looking for is either

$$\begin{aligned} x_{2^{j+1}+2^j+2^{j-1}-2} &= \cos(\sin(\sin\cdots(\sin(\cos(\sin 1)))\cdots)) \\ &\geq \cos(\sin(\sin(\sin(\cos(\sin 1)))))) \\ &> 0.85 \end{aligned}$$

or

$$x_{2^{j+1}+2^j-3} = \sin(\cos(\sin\cdots(\sin(\cos 1))\cdots)) < \sin 1 < 0.85\,;$$

and the former is the larger. Likewise, the choice for second smallest is between

$$x_{2^{j+1}+2^j-4} = \cos(x_{2^j+2^{j-1}-2}) > \cos 1 > 0.54$$

and

$$x_{2^{j+1}+2^j+2^{j-1}-1} = \sin(x_{2^j+2^{j-1}+2^{j-2}-1}) < \sin(\sin(\sin(\sin(\cos(\sin 1)))))) < 0.53\,;$$

and the latter is the smaller. Finally, we calculate the second largest element of X_{140}: it is $x_{2^j+2^{j-1}+2^{j-2}-2}$ with $j = 140$. This is an expression with 137 sine terms in the middle,

$$x_{2^{140}+2^{139}+2^{138}-2} = \cos(\sin(\sin\cdots(\sin(\cos(\sin 1)))\cdots)) = 0.989717\cdots,$$

which is less than 0.99. Therefore, the element we found above is the only element up to X_{140} which exceeds 0.99, and it is x_n with

$$n = 2^{140} + 2^{139} - 2 = 2090694862362245919518973588060783891185662\,.$$

Comment. This may be a surprisingly large value of n, seeing that 0.99 is not *very* close to 1. Suppose that you tried to brute–force this question by calculating $x_2, x_3, x_4, \ldots$ sequentially until you obtained an answer greater than 0.99. With advanced numerical software it is currently possible to perform about 2^{30} sine and cosine calculations per second; at this rate, solving the problem by brute force would take something like $2^{86} \approx 10^{26}$ years.

Q1675 Prove that if a, b, c are positive numbers, then

$$\frac{4a+3b+2c}{4c+3b+2a} + \frac{4b+3c+2a}{4a+3c+2b} + \frac{4c+3a+2b}{4b+3a+2c} \geq 3\,.$$

SOLUTION First note that, for any positive x and y,

$$\frac{x}{y} + \frac{y}{x} \geq 2\,.$$

There are many easy ways to prove this. Applying this to the three denominators in the required expression, taken in pairs, we have

$$\frac{4a+3c+2b}{4c+3b+2a}+\frac{4c+3b+2a}{4a+3c+2b}\geq 2\,,$$
$$\frac{4b+3a+2c}{4a+3c+2b}+\frac{4a+3c+2b}{4b+3a+2c}\geq 2\,,$$
$$\frac{4c+3b+2a}{4b+3a+2c}+\frac{4b+3a+2c}{4c+3b+2a}\geq 2\,.$$

Now add these three expressions, combining the fractions with the same denominator, to get

$$\frac{7a+6b+5c}{4c+3b+2a}+\frac{7b+6c+5a}{4a+3c+2b}+\frac{7c+6a+5b}{4b+3a+2c}\geq 6\,. \tag{$*$}$$

We seek to write the numerator of the first fraction in the form

$$7a+6b+5c=\alpha(4a+3b+2c)+\beta(4c+3b+2a)\,,$$

because then, after dividing by $4c+3b+2a$, the first term will give a fraction from our required expression, and the second will simply give a constant. Equating coefficients of a,b,c, we want

$$4\alpha+2\beta=7\,,\quad 3\alpha+3\beta=6\,,\quad 2\alpha+4\beta=5\,;$$

it is easy to solve these equations to get $\alpha=\frac{3}{2}$ and $\beta=\frac{1}{2}$. (Although we have three equations in two variables, any one is implied by the others, so there are only two independent equations.) Hence,

$$\frac{7a+6b+5c}{4c+3b+2a}=\frac{3}{2}\Big(\frac{4a+3b+2c}{4c+3b+2a}\Big)+\frac{1}{2}\,;$$

doing something similar for the other two fractions in $(*)$ yields

$$\frac{3}{2}\Big(\frac{4a+3b+2c}{4c+3b+2a}+\frac{4b+3c+2a}{4a+3c+2b}+\frac{4c+3a+2b}{4b+3a+2c}\Big)+\frac{3}{2}\geq 6$$

and hence

$$\frac{4a+3b+2c}{4c+3b+2a}+\frac{4b+3c+2a}{4a+3c+2b}+\frac{4c+3a+2b}{4b+3a+2c}\geq 3\,.$$

Alternative solution, submitted by Hyunbin Yoo, South Korea. We shall use the Cauchy–Schwartz Inequality

$$(u_1v_1+u_2v_2+u_3v_3)^2\leq(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)\,.$$

In particular, if we take $u_k=\sqrt{x_k}$ and $v_k=\sqrt{y_k}$, then this becomes

$$(x_1+x_2+x_3)(y_1+y_2+y_3)\geq\big(\sqrt{x_1y_1}+\sqrt{x_2y_2}+\sqrt{x_3y_3}\big)^2\,.$$

First, we denote the left hand side of the requested inequality by L, and we add 1 to each fraction so that they all share the same numerator. Thus,

$$\begin{aligned} L+3 &= \frac{6a+6b+6c}{4c+3b+2a} + \frac{6a+6b+6c}{4a+3c+2b} + \frac{6a+6b+6c}{4b+3a+2c} \\ &= 6(a+b+c)\left(\frac{1}{4c+3b+2a} + \frac{1}{4a+3c+2b} + \frac{1}{4b+3a+2c}\right). \end{aligned}$$

Notice that the sum of the denominators is equal to $9(a+b+c)$. The previous line can therefore be written as

$$\begin{aligned} L+3 = \frac{6}{9}\big((4c+3b+2a) + (4a+3c+2b) + (4b+3a+2c)\big) \\ \times\left(\frac{1}{4c+3b+2a} + \frac{1}{4a+3c+2b} + \frac{1}{4b+3a+2c}\right). \end{aligned}$$

Now we can apply the Cauchy–Schwarz inequality stated above, taking x_1, x_2, x_3 to be the three terms in the first bracket and y_1, y_2, y_3 to be the three fractions in the second bracket. Since $y_k = 1/x_k$ in each case, we have

$$L+3 = \frac{2}{3}(x_1+x_2+x_3)(y_1+y_2+y_3) \geq \frac{2}{3}\big(\sqrt{x_1y_1} + \sqrt{x_2y_2} + \sqrt{x_3y_3}\big)^2 = 6$$

and so $L \geq 3$.

Comment. The Cauchy–Schwartz Inequality is very important in many areas of advanced algebra and calculus. It can be generalised to any number of variables, so that

$$(u_1v_1 + \cdots + u_nv_n)^2 \leq (u_1^2 + \cdots + u_n^2)(v_1^2 + \cdots + v_n^2).$$

A **different solution**, using Muirhead's Inequality, was received from Soham Dutta, DPS Ruby Park, India. This will be the subject of an article by Soham which is scheduled to appear in the next issue of *Parabola*.

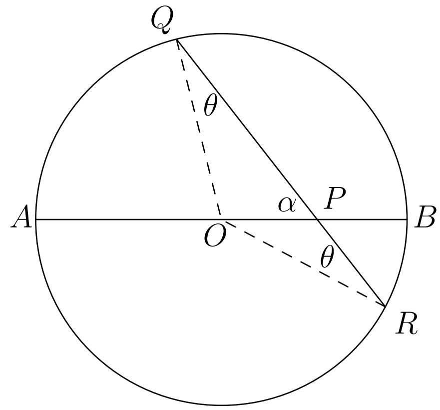
Q1676 Let C be a circle on diameter AB, and let α be an acute angle. For any point P on AB, draw a chord QR of the circle passing through P, with the angle between AB and QR being equal to α. Given that the quantity

$$|PQ|^2 + |PR|^2$$

is the same for all choices of P, determine the angle α; and then find $|PQ|^2 + |PR|^2$ in terms of the radius of C.

SOLUTION Let O be the centre of the circle, let r be the radius of the circle, and write

$\angle OQP = \angle ORP = \theta$. Then $\angle POQ = 180° - \alpha - \theta$ and $\angle POR = \alpha - \theta$.



Apply the sine rule to triangle OPQ to get

$$\frac{PQ}{\sin(180° - \alpha - \theta)} = \frac{r}{\sin \alpha}$$

and hence

$$|PQ| = r\,\frac{\sin(180° - \alpha - \theta)}{\sin \alpha} = r\,\frac{\sin(\alpha + \theta)}{\sin \alpha} = r\,\frac{\sin \alpha \cos \theta + \cos \alpha \sin \theta}{\sin \alpha}\,.$$

Treating triangle ORQ similarly gives

$$|PR| = r\,\frac{\sin(\alpha - \theta)}{\sin(180° - \alpha)} = r\,\frac{\sin \alpha \cos \theta - \cos \alpha \sin \theta}{\sin \alpha}\,;$$

therefore,

$$\begin{aligned}|PQ|^2 + |PR|^2 &= \frac{r^2}{\sin^2 \alpha}\left(2\sin^2 \alpha \cos^2 \theta + 2\cos^2 \alpha \sin^2 \theta\right)\\ &= 2r^2\Big(\cos^2 \theta + \frac{\cos^2 \alpha}{\sin^2 \alpha}\sin^2 \theta\Big)\\ &= 2r^2\Big(1 + \Big(\frac{\cos^2 \alpha}{\sin^2 \alpha} - 1\Big)\sin^2 \theta\Big)\,.\end{aligned}$$

Now if P changes, then θ changes and therefore the above quantity changes, unless we have that $(\cos^2 \alpha)/(\sin^2 \alpha) - 1 = 0$; that is, $\alpha = 45°$. This is the only case in which $|PQ|^2 + |PR|^2$ has a constant value, and that value is

$$|PQ|^2 + |PR|^2 = 2r^2\,.$$

Another solution was submitted by Hyunbin Yoo, South Korea, who took M to be the midpoint of QR and showed that

$$\begin{aligned}|PQ|^2 + |PR|^2 &= (|OM| + |MP|)^2 + (|OM| - |MP|)^2\\ &= 2\big(r^2 + d^2(\cos^2 \alpha - \sin^2 \alpha)\big)\,.\end{aligned}$$

For this to be constant, we need $\cos^2 \alpha - \sin^2 \alpha = 0$, which gives $\alpha = 45°$ and $|PQ|^2 + |PR|^2 = 2r^2$.

Q1677 For this question, a *knockout contest* among n entrants means the following. Let k be the integer for which $2^{k-1} < n \le 2^k$. Then $2^k - n$ players (chosen at random) are "given a bye" in the first round: that is, they progress to the next round without playing a match. The remaining $2n - 2^k$ players play in pairs (once again chosen at random), and the $n - 2^{k-1}$ winners also progress. This leaves 2^{k-1} entrants who will play in pairs, leaving 2^{k-2} winners in the next round; and so on; until the overall winner is decided by a match between the last 2 players. Note that there are no byes after the first round, and so the eventual winner will play either $k-1$ or k matches, depending on whether they do or do not receive a bye in the first round.

(a) If there are n players at the beginning of the competition, then how many matches will be played altogether?
Comment: this is a well known problem and there is a very easy solution. Do not try to consider the number of byes, the number of rounds or other details!

(b) A football club wishes to rank the three strongest arm–wrestlers from a pool of 100 candidates by using a knockout contest to determine the best arm–wrestler; then another knockout contest to determine the second–best; then another to determine the third–best. Show that if the contests are carefully organised, then this can be accomplished in 113 matches altogether; but that fewer than 105 matches will never be enough.

SOLUTION To determine the winner from n entrants, $n-1$ of them must lose a match. Therefore, $n-1$ matches will be played. This answers question (a).

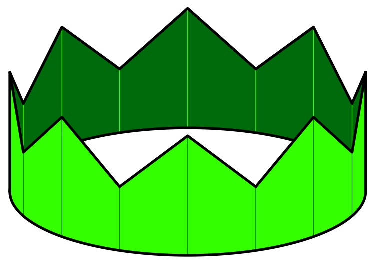
For (b), this shows immediately that 99 matches will be required in the first contest. Now, the best contestant will have played either 6 or 7 matches in the first round; and the second–best contestant must have been the loser in one of these matches (because nobody except the best can defeat the second–best). Therefore, there is no need to have all 99 unranked competitors in the second contest, but only these 6 or 7; there will be 5 or 6 matches; and the final winner of this contest will be the second–best arm–wrestler overall.

Now consider what happened to the third–best entrant in the first contest. That entrant must have been beaten by either the best or second–best entrant.

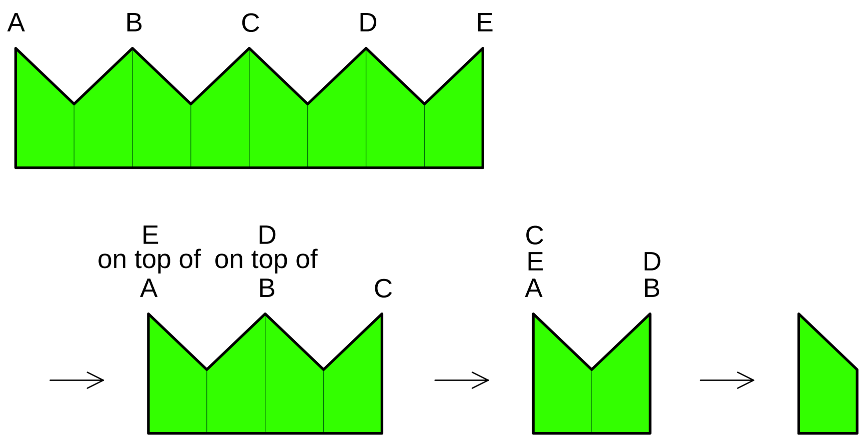
- Those in the first category went on to play in the second contest, and the third–best player must have been beaten by the second–best. But since there were 6 or 7 entrants in this contest, the second–best defeated 2 others (and had a bye in the first round), or 3 others. Only these 2 or 3 are possible candidates for the third–best player.
- Now we consider those in the second category. In the first contest, the second–best player may have played 1 game only (losing to the best player in the first round); or 7 games (not receiving a bye in the first round, and eventually losing to the best player in the final); or anywhere in between. So the number of players defeated by the second–best player in the first contest is from 0 to 6, and these are the remaining candidates for third–best arm–wrestler.

Therefore, the total number of competitors in the contest for third place is from 2 to 9; and the number of matches played will be from 1 to 8. Putting all this information together, the best three arm–wrestlers can always be found in $99+6+8 = 113$ matches; and the minimum requirement is $99 + 5 + 1 = 105$ matches.

Q1678 Last Christmas, I pulled a Christmas cracker, and out popped the traditional paper crown.



While inside the cracker it had been flattened out between two opposite points A and E, and then folded right half over left three times, as in the diagrams.



It's clear that if the crown is unfolded, then some of the creases that have been made will point towards the outside of the crown, and some will point towards the inside. Is it possible to now refold the crown in the same way as before, but starting with a *different* pair of opposite points instead of A and E, and *without* reversing any of the folds already made?

SOLUTION Begin by cutting the crown at A and unfolding it into a long strip with E in the middle. Then fold it back again. The fold at E will point outwards: we denote this by O. Now given any pre–existing folds, performing another right–on–left fold will give the following result. The new folds will begin with a copy of the folds we have already, because it is just like starting with a strip of half the length and doing the same as has already been done. Then there will be an O fold, because this will always be the case for the middle fold. Then there will be the folds we have done already, but taken in reverse order; and with outward folds changed to inward and *vice versa*. The operation of performing one extra fold on an existing fold can be represented as

$$w \;\mapsto\; w\,O\,w^*\,,$$

where w^* means the string of symbols obtained from w by reversing the order of the symbols and swapping their identity. We know that we start with just O; folding three times gives

$$\begin{aligned} O &\mapsto O(O)I \\ &\mapsto OOI(O)OII \\ &\mapsto OOIOOII(O)OOIIOII\,. \end{aligned}$$

(The brackets have been inserted to identify the new middle element in each line, but do not have any real significance.) Now stick the crown together again at A; this will give an O fold, and so the complete circle of folds, starting at A, is

$$OOOIOOIIOOOIIOII\,.$$

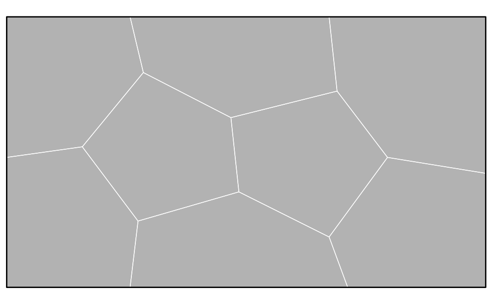
To solve the problem, notice that the I folds always occur in pairs, with one exception of a single I between two Os. If we were to start the folding process at a different place, then this isolated I would end up in a different place; and so either it, or one of its adjacent Os, would not match the folds we have already. Therefore, it is impossible to fold the crown from a different starting point without reversing any existing folds.

Q1679 David is designing a tiling pattern for his rectangular bathroom floor. Most people have tilings which consist of rectangles or hexagons, but David thinks this is boring, so he has decided to use pentagons. Any sorts of pentagonal shapes are acceptable: they do not have to be regular pentagons, and they need not all be congruent. Moreover, David has decided that an attractive design should have three further features:

- each corner of the rectangle should belong to only one pentagon, and no two corners can belong to the same pentagon;
- each boundary point of the rectangle should belong to at most two pentagons;
- each interior point of the rectangle which is a corner of a pentagon should belong to exactly two other pentagons. (*This was stated inaccurately in the previous issue*.)

Show that in order to fulfil these requirements, David must use exactly 8 pentagons; and draw a possible design for the bathroom floor.

SOLUTION It is not hard to find by trial and error a possible design with 8 pentagons.



The difficult part of the problem is to prove that no satisfactory design is possible with more or fewer than 8 pentagons. This can be done by treating it as a problem in *graph theory*. In a diagram such as the above, or a hypothetical other diagram satisfying the design brief, the lines on and inside the rectangle are referred to as **edges**. The points where two or more edges meet, including the corners of the rectangle, are called **vertices** (one is called a **vertex**). The pentagons are called the **regions** of the graph; the infinite area surrounding the rectangle is also counted as a region. There are clearly 4 pentagons at the corners of the rectangle; let the number of pentagons touching the boundary, other than these 4, be m, and let the number which do not touch the boundary be n. We shall count the number of regions, vertices and edges in an acceptable design.

- The number of regions is easy: there are $n + m + 4$ pentagons and the external region, in total
$$(\text{number of regions}) = n + m + 5\,.$$
- To count the edges, we note that each pentagon is surrounded by 5 edges, giving $5n + 5m + 20$. This counts every internal edge twice, and every boundary edge once. The number of boundary edges is 2 for each "corner" pentagon and 1 for each "boundary" pentagon, a total of $m + 8$. If we add this to the previous figure, every edge will have been counted twice, and so
$$2(\text{number of edges}) = 5n + 6m + 28\,.$$
- To count the number of vertices we take 5 for each pentagon. This counts "internal" vertices 3 times each, boundary vertices twice and corner vertices once. By adding the $m + 8$ boundary and corner vertices, and adding again the 4 corner vertices, we end up counting every vertex three times. Therefore,
$$\begin{aligned}3(\text{number of vertices}) &= 5(n + m + 4) + (m + 8) + 4 \\ &= 5n + 6m + 32\,.\end{aligned}$$

Since the design is drawn on a flat surface without edges crossing, a very important result known as **Euler's Theorem for planar maps** states that
$$(\text{number of regions}) + (\text{number of vertices}) = (\text{number of edges}) + 2\,.$$

Multiplying by 6 in order to avoid fractions, and substituting the counts we have just calculated, gives
$$6(n + m + 5) + 2(5n + 6m + 32) = 3(5n + 6m + 28) + 12\,,$$
which miraculously simplifies to $n = 2$. So the design must include exactly 2 internal pentagons.

To determine the number m of boundary pentagons, we concentrate on the internal vertices of the design – those that do not lie on the boundary of the rectangle. Each boundary pentagon gives one of these by itself, and two others shared with another boundary or corner pentagon; each corner pentagon gives two internal vertices, shared with another pentagon. Hence,

$$\text{(number of internal vertices)} = m + \frac{2(m+4)}{2} = 2m + 4\,.$$

But as there are 2 internal pentagons, and noting that they may share vertices, we have

$$5 \le 2m + 4 \le 10$$

and so $m = 1, 2$ or 3. Finally, consider the number of internal edges that connect these internal vertices (not counting edges which go to the boundary). We count 3 for each of the vertices from the boundary pentagons, and 2 for each of the "shared" vertices; but this counts every edge twice; so

$$2\text{(number of edges)} = 3m + 2(m+4)\,.$$

It follows from this that m is even; so of our previous possibilities, only $m = 2$ remains. Therefore, the bathroom floor must consist of 2 internal pemtagons, 2 boundary pentagons and 4 corner pentagons, a total of 8.

Q1680 Calculate the limit of the sum

$$S_n = \sum_{k=0}^{n-1} \frac{n}{(n+k)^2}$$

as n tends to ∞.

SOLUTION We have

$$\begin{aligned}
S_n &= \sum_{k=0}^{n-1} \frac{n}{(n+k)^2} \\
&> \sum_{k=0}^{n-1} \frac{n}{(n+k)(n+k+1)} \\
&= \sum_{k=0}^{n-1} \Bigl(\frac{n}{n+k} - \frac{n}{n+k+1}\Bigr) \\
&= \Bigl(\frac{n}{n} - \frac{n}{n+1}\Bigr) + \Bigl(\frac{n}{n+1} - \frac{n}{n+2}\Bigr) + \cdots + \Bigl(\frac{n}{2n-1} - \frac{n}{2n}\Bigr) \\
&= \frac{n}{n} - \frac{n}{2n} \\
&= \frac{1}{2}
\end{aligned}$$

and similarly

$$\begin{aligned}
S_n &< \sum_{k=0}^{n-1} \frac{n}{(n+k-1)(n+k)} \\
&= \frac{n}{n-1} - \frac{n}{2n-1} = \frac{1}{2} + \frac{1}{n-1} - \frac{1}{4n-2}\,.
\end{aligned}$$

Putting these inequalities together,

$$\frac{1}{2} < S_n \le \frac{1}{2} + \frac{1}{n-1} - \frac{1}{4n-2}\,.$$

Now if $n \to \infty$, then the expression on the right–hand side approaches $\frac{1}{2}$. Since S_n is trapped between this quantity and $\frac{1}{2}$, it also must approach a limit of $\frac{1}{2}$.

Comment. For readers who have studied integration and Riemann sums, the problem can also be solved by showing that S_n is a Riemann sum for $f(x) = x^{-2}$ on the interval $1 \le x \le 2$, and hence

$$\lim_{n\to\infty} S_n = \int_1^2 \frac{dx}{x^2} = \frac{1}{2}\,.$$

This approach was pointed out by Hyunbin Yoo, South Korea.