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Finding e in a uniform distribution

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1 Introduction

Euler's number e can be represented in a variety of ways. It was discovered by Bernoulli as the sequence limit (see [1])

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

and can also be represented as continued fractions, as infinite products, and as infinite series, of which the simplest is the Maclaurin series

$$e = \sum_{n=0}^{\infty}\frac{1}{n!}\,.$$

There exist different ways to write e as an infinite series, many of which have been obtained by combining the terms of the Maclaurin series as in [2, 3, 4]. In this paper, we express e as an infinite series in a new manner, from a recently discovered relation between the uniform and the exponential probability distributions. We also provide a direction which could lead to the discovery of new representations of Euler's number.

2 Exponential distributions from uniform distributions

The exponential probability distribution is the only distribution that presents a constant hazard rate when describing a distribution of temporal intervals. Illustrated in Figure 1, this property relates to the fact that in an exponential distribution of intervals, the relative likelihood of an interval with duration t decreases at the same rate as the probability that an interval lasts longer than t seconds, as t increases.

In experimental psychology, the exponential distribution is sometimes used to schedule food with the purpose of making this reward random in time and therefore unpredictable for animal subjects. Yet, because the exponential distribution relies on an infinite interval range, its use poses the operational problem of long, potentially infinite, intervals which forces the use of approximations by experimenters. But approximations of an exponential distribution only offer approximations of the desired mathematical property of constant hazard rate.

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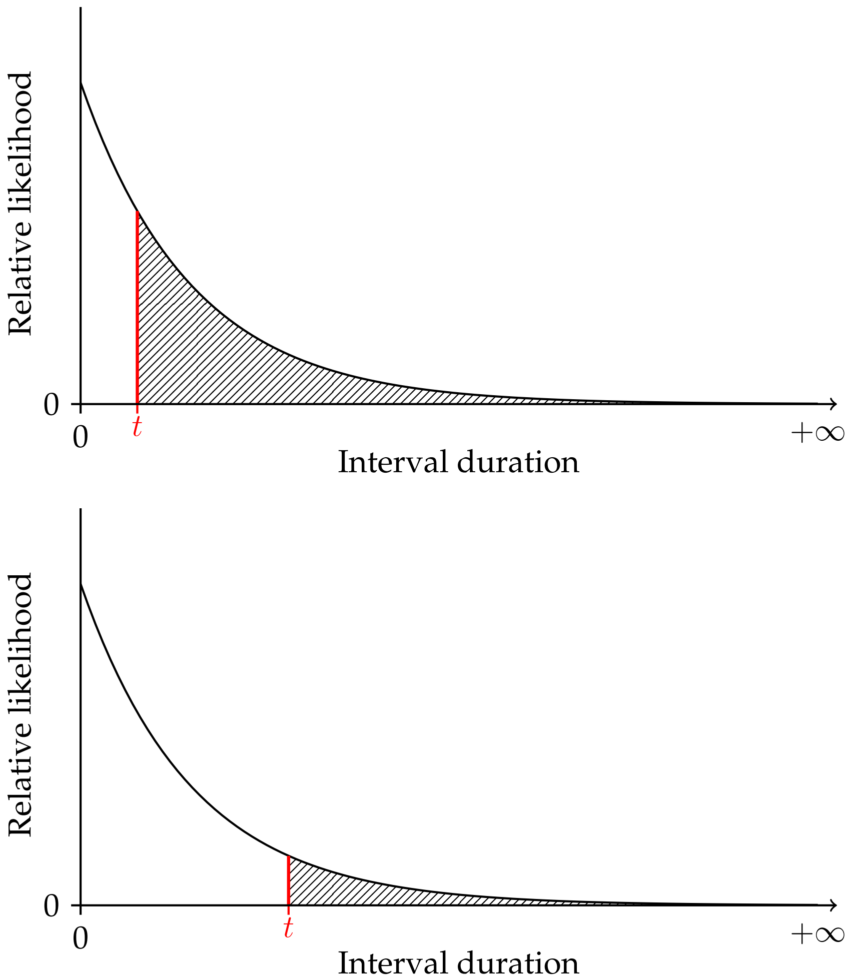

Figure 1: Constancy of the ratio between the relative likelihood of an interval of duration t and the probability that an interval lasts longer than t, in an exponential distribution of temporal intervals.

Looking for a way to circumvent this problem, Bugallo, Machado and Vasconcelos [5] found that certain combinations of a distribution of intervals ending with the reward and a distribution of intervals ending without the event of interest (empty intervals) allowed a constant hazard rate of reward on a finite interval range.

Figure 2 illustrates an example of such combination wherein the uniform overall distribution of intervals (with mean interval duration T) contains a sub-distribution of rewarded intervals with triangular shape and a complementary sub-distribution of empty intervals with trapezoidal shape. It is easy to see in Figure 2 that both the relative likelihood that a rewarded interval lasts t seconds (the red height), and the probability that an interval from the overall distribution lasts longer that t (hatched area), decrease linearly from a maximum at $t = 0$ to 0 at $t = 2T$, and by there maintain constant the hazard rate from 0 to $2T$.

On the basis of a uniform overall distribution, the method allows a constant hazard

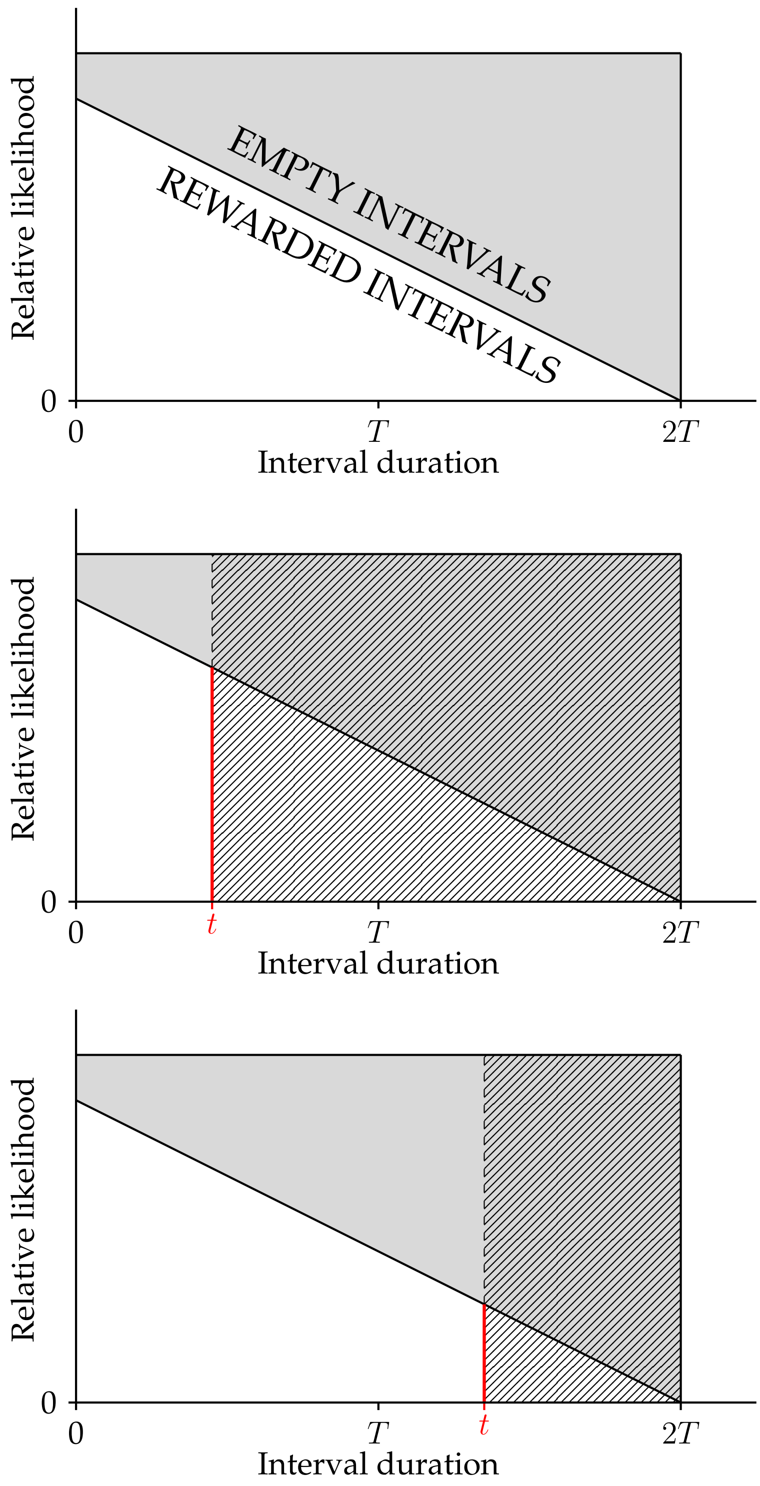

Figure 2: Example of a combination of a distributions that maintain constant hazard rate on a finite interval range. Constancy of the ratio between the relative likelihood of an interval of duration t and the probability that an interval lasts longer than t, in an exponential distribution of temporal intervals.

rate as long as the sub-distribution of rewarded intervals has a right-triangular shape, with right angle on the origin of axis and with a side matching the range of definition of the overall distribution. That is, the mass p of the sub-distribution of rewarded intervals can vary between 0, excluded, and 0.5, for which value the hypotenuse of the right-triangular sub-distribution matches the diagonal of the overall rectangular distribution (in this latter case the sub-distribution of the empty intervals becomes the symmetrical of the sub-distribution of rewarded intervals). Once the parameters $2T$ and p are set, intervals of the combination of distributions are obtained by drawing intervals from a uniform distribution defined between 0 and $2T$, which intervals are then characterized as rewarded or empty according to a random process whose probability outcome matches the proportions of the relative likelihood in the two sub-distributions at the duration corresponding to the drawn interval (see [5] for detailed explanations).

Interestingly, Bugallo, Machado and Vasconcelos [5] showed that if intervals are independently drawn, one by one, according to a random variable X with uniform distribution, with mean T, and are characterized as rewarded or empty according to the combination method (with a probability p of rewarded intervals) and are laid end to end until the occurrence of a reward, then the resulting intervals form a random variable Y that is exponentially distributed (with rate parameter p/T). From this relation we will now determine e.

3 An new expression for e

Let us consider the function $f : [0; \infty) \to [0; \infty)$ by

$$f(t) = \lambda e^{-\lambda t}$$

and let Y be a random variable with f as density function. Then Y is exponentially distributed, and

$$P(Y \leq t) = \int_0^t f(s)\, ds = \int_0^t \lambda e^{-\lambda s}\, ds = 1 - e^{-\lambda t}\,.$$

Then

$$P(Y > t) = e^{-\lambda t}$$

which is the probability that an interval of the initial exponential distribution is longer than t. This probability can also be obtained by adding, for $n = 1, 2, \ldots$, the products of the probability $P(N = n)$ that an interval from the random variable Y is made of n intervals from the random variable X, where N is the number of X intervals in a Y interval, with the conditional probability $P(Y > a \,|\, N = n)$ that an interval from the random variable Y is longer than a given that it is made of n intervals from the random variable X. Thus, as

$$e^{-\lambda a} = \sum_{n=1}^{\infty} P(N = n) P(Y > a \,|\, N = n)\,,$$

we have

$$e = \left[\sum_{n=1}^{\infty} P(N = n) P(Y > a \,|\, N = n) \right]^{-\dfrac{T}{ap}}$$

where p is the unconditional probability that an X interval is rewarded.

Intervals of the random variable X are drawn randomly, one after the other, and characterized as rewarded or empty according to the combination method explained in the previous section. A rewarded interval always has a probability p to be drawn at each round. Therefore, the probability associated to the number of intervals from the random variable X in an interval of the random variable Y is defined by a geometric distribution

$$P(N = n) = p(1 - p)^{n-1} \,.$$

Now, let us consider the simple case in which $2T = 1$, $p = 0.5$ and $a = 1$. The formula for e simplifies to

$$e = \left[\sum_{n=1}^{\infty} \frac{1}{2^n} P(Y > 1 \,|\, N = n) \right]^{-1} .$$

Now, $P(Y > 1 \,|\, N = n)$ corresponds to the sum of probabilities that the sum of the empty intervals is greater than 1, and of the probability that the interval of random variable Y is greater than 1 and the sum of the empty intervals is smaller than 1. We then have

$$P(Y > 1 \,|\, N = n) = P\left(\sum_{m=1}^{n-1} x_m > 1 \right) + P\left((Y > 1) \cap \sum_{m=1}^{n-1} x_m < 1 \right)$$

where x_{n-1} is the interval length of the last empty interval. Therefore,

$$P(Y > 1 \,|\, N = n) = P\left(\sum_{m=1}^{n-1} x_m > 1 \right) + P\left(\sum_{m=1}^{n-1} x_m < 1 \right) P\left(Y > 1 \middle| \sum_{m=1}^{n-1} x_m < 1 \right) .$$

We first consider the probability that the sum of the empty intervals is smaller than 1. The probability that the first empty interval is smaller than 1 is 1, since maximal interval length, $2T$, is 1. Then, to find the probability that the sum of the two first empty intervals is smaller than 1, we calculate the probability that this sum is greater than 1, which we will subtract to 1. For the sum of the two first empty intervals to be greater than 1, the length t obtained from the distribution of the first empty interval (the same for any empty interval: $w(t) = 2t$; see the top graph in Figure 3), must be associated to a second interval of a length greater than $1 - t$. Within the distribution of the second empty interval on 0 to 1, we have to take at any t the proportion of intervals that would be longer than 1 if associated to a first empty interval of this certain t length. That is,

$$P(x_1 + x_2 > 1) = \int_0^1 \left(2t \int_{1-t}^1 2x \, dx \right) dt \,,$$

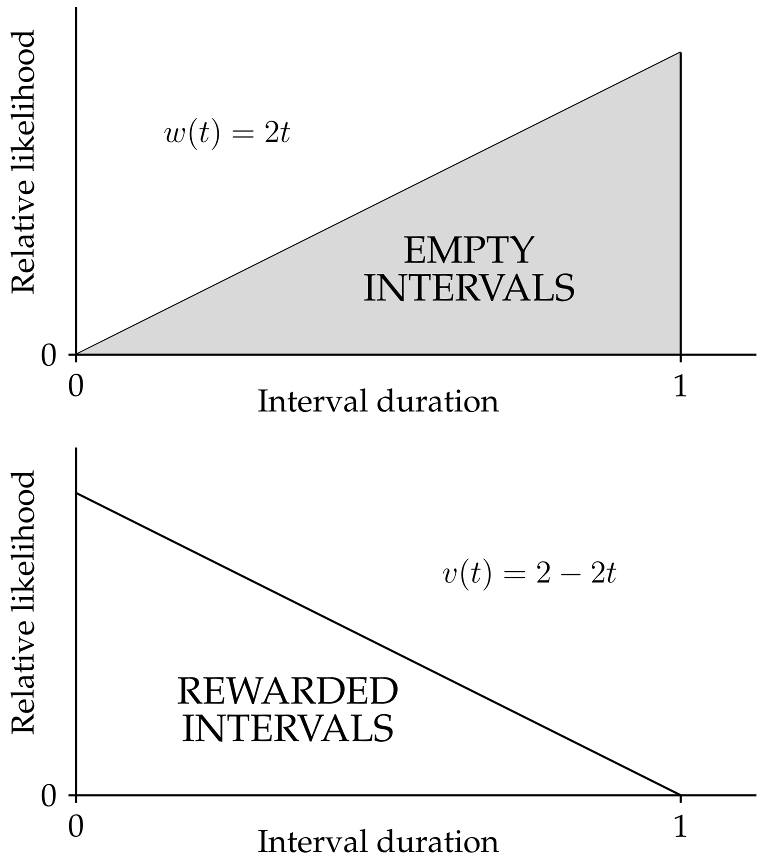

Figure 3: Individual density functions of rewarded and empty intervals.

where x_1 and x_2 are the durations of the first and second empty intervals, respectively. Thus, we have

$$P(x_1 + x_2 < 1) = 1 - \int_0^1 \left(2t \int_{1-t}^1 2x\, dx \right) dt\,.$$

To obtain the probability that the sum of k empty intervals is smaller than 1, we can multiply the probability that the sum of $k-1$ empty intervals is smaller than 1 by the probability that the adding of the kth empty interval would still not make 1 when the sum of the previous $k-1$ empty intervals was smaller than 1.

On the interval $[0;1]$, the distribution of the summed empty intervals corresponds to the shape of a power of t; at any point of the area of the previous distribution of the summed empty intervals, on the interval $[0;1]$, each new empty interval comes to add its own distribution of intervals. Thus, we find the density function of the intervals formed by the sum of $k > 1$ empty intervals on the interval $[0;1]$ is

$$z(t) = \frac{t^{2k-3}}{\int_0^1 x^{2k-3}\, dx}\,.$$

We see that

$$P\left(\sum_{j=1}^{k} x_j < 1\right) = P\left(\sum_{j=1}^{k-1} x_j < 1\right)\left(1 - \int_0^1 \left(\frac{t^{2k-3}}{\int_0^1 x^{2k-3}\,dx}\int_{1-t}^{1} 2x\,dx\right) dt\right)$$
$$= P\left(\sum_{j=1}^{k-1} x_j < 1\right)\frac{1}{2k^2 - k}\,.$$

Now, with

$$P\left(x_1 < 1\right) = 1\,,$$

we see that

$$P\left(\sum_{j=1}^{k} x_j < 1\right) = \frac{1}{\prod_{j=1}^{k}(2j^2 - j)} = \frac{2^k}{(2k)!}\,.$$

Replacing k by n to consider all intervals (with $k = n - 1$), we have

$$P\left(\sum_{m=1}^{n} x_m < 1\right) = \frac{2^{n-1}}{(2n-2)!}\,.$$

We now consider the probability that the sum of all intervals, including the rewarded interval that ends with the event of interest, is greater than 1, given that the sum of the empty intervals is smaller than 1. Pictured within the distribution of the rewarded intervals defined by $v(t) = 2 - 2t$ (see bottom graph in Figure 3), it corresponds at any t to the proportion of rewarded intervals that will be greater than 1 when summed with an interval from the distribution of the sum of the empty intervals. That is,

$$P\left(Y > 1 \middle| \sum_{j=1}^{k} x_j < 1\right) = \int_0^1 \left((2 - 2t)\int_{1-t}^{1} \frac{x^{2k-1}}{\int_0^1 x^{2k-1}\,dx}\,dx\right) dt = \frac{k}{k+1} = \frac{n-1}{n}\,.$$

Thus, we have all elements to obtain the probability that an interval from the random variable Y is longer than 1 given that it is made of n intervals from the random variable X. We have,

$$P(Y > 1 \,|\, N = n) = \left(1 - \frac{2^{n-1}}{(2n-2)!}\right) + \frac{2^{n-1}}{(2n-2)!}\frac{n-1}{n} = 1 - \frac{2^{n-1}}{n(2n-2)!}\,.$$

Finally, we assemble the formula for e:

$$e = \left[\sum_{n=1}^{\infty} \frac{1}{2^n}\left(1 - \frac{2^{n-1}}{n(2n-2)!}\right)\right]^{-1} = \left[\sum_{n=1}^{\infty}\frac{1}{2^n} - \sum_{n=1}^{\infty}\frac{2n-1}{(2n)!}\right]^{-1}$$

which simplifies to

$$e = \left[1 - \sum_{n=1}^{\infty}\frac{2n-1}{(2n)!}\right]^{-1}\,.$$

When $n = 0$,

$$\frac{2n-1}{(2n)!} = -1\,,$$

so we have

$$e = -\left[\sum_{n=0}^{\infty} \frac{2n-1}{(2n)!}\right]^{-1} \,.$$

We recognize here the formula [4, Equation (9)] which was obtained by recombination of the terms of the Maclaurin series using a compressing technique:

$$e = \left[\sum_{n=0}^{\infty} \frac{1-2n}{(2n)!}\right]^{-1} \,.$$

Other representations of e may be obtained by solving

$$e = \left[\sum_{n=1}^{\infty} p(1-p)^{n-1} P(Y > a \,|\, N = n)\right]^{-\dfrac{T}{ap}}$$

that may not have yet been discovered with recombination of the Maclaurin series.

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