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The Axiom of Choice

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1 Introduction

What is choice in mathematics? It is quite natural in our everyday lives; we make the choice of what to eat for breakfast, where to spend the day, and what time we want to fall asleep. We have a finite amount of choices to make in our lives, and it is trivial to say that we can make those choices. However, what about in mathematics? Can we make infinite choices using logic and numbers without consequence? In this paper, I will outline an understanding of the *Axiom of Choice*. I will then delve into how to justify an axiom, and apply those principles to the Axiom of Choice. I will also explain various controversial results that arise from its usage, and how they are not so nonsensical after all. Finally, I will discuss alternate versions of the Axiom of Choice and argue that we should prefer the weakest conception of it as possible. Before reading this article, please refer to the introduction article *Set Theory as the Foundation of Mathematics with Focus on the Axiom of Choice*, also in this issue of *Parabola*, which illustrates how set theory provides a foundation for mathematics; this is necessary historical and mathematical background to understanding axioms, the justification of axioms, and from where this debate has evolved. For further reading, please refer to the article *Set Theory as the Foundation of Mathematics* by Rida Naveed Ilahi.

2 The Axiom of Choice

Choice and Choice Functions

A *choice function* is a function f such that for all non-empty sets within a set A, $f(A') \in A'$ when $A' \in$ A. In essence, this allows us to map any set to an element that it contains. For example, in the set $A = \big\{\{1\}, \{2, 4, 6\}\big\}$, some choice function can map $\{1\} \to 1$ and $\{2, 4, 6\} \to 4$. The choice function chooses an element from each set within A. If we denoted a new set B formed of the elements that the choice function chose ($B = \{1, 4\}$), then B is called a *choice set*. For a more concrete example, suppose I was walking through my school, choosing one person from each classroom to represent that class and writing down each choice I made. My paper containing my choices is my choice function, and this new group of students I chose is my choice set. For finite sets then, choice functions are simple - we can explicitly map each non-empty subset to an element of each. For infinite sets, the problem becomes more complicated.

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The *Axiom of Countable Choice* states that there exists a choice function for all countably infinite sets. One example is of the set of all the non-empty subsets of the natural numbers. A simple choice function can map each subset to the lowest element it holds [8]. For example, the subset $\{1, 2, 3\}$ maps to 1, and $\{4, 17, 23\}$ maps to 4.

The same cannot be said about uncountably infinite sets, as no one has proved that choice functions exist for all such sets. Indeed, some model theoretic arguments imply that, for some such sets, no well-defined choice function can be found [8].

Axiom of Choice

The *Axiom of Choice* states that every set has a choice function, regardless of the cardinality of the set. It is the last axiom of the ZFC system (and is represented by the "C" in ZFC to indicate its inclusion).

To discuss whether the inclusion of the Axiom of Choice in ZFC is valid, I will first discuss why it is an axiom. Gődel proved that the ZF axioms are consistent with the Axiom of Choice if ZF is consistent [4], reducing the problem to depend on Gődel's Second Incompleteness Theorem and showing that the axioms of ZF cannot be used to prove the negation of the Axiom of Choice. In 1963, Cohen proved the Axiom's independence from ZF [3], showing that the axioms of ZF and the negation of the Axiom of Choice is a consistent system, which are two obviously contradictory results. Thus, this axiom is independent from ZF, and is fit to be called an axiom. As such, the justifications that I will be discuss pertain to its unique nature as an axiom (as opposed to a theorem).

Types of Motivation for its Inclusion

Before I begin to delve into the details of the Axiom of Choice, we must understand how axioms are justified. As the basis of our mathematical systems, we cannot mathematically prove the "truth" of an axiom using some other mathematical object; we can only understand the consistency and soundness of such a system based on the axioms. Proofs will not work in this context, so justifications are naturally the next-most powerful tools for understanding the arguments for and against the inclusion of axioms in our mathematical systems. As a result, I will use the types of justifications Gődel outlined in his 1964 paper, *What is Cantor's Continuum Hypothesis*?

The first form of justification Gődel outlines is *intrinsic justification*, which are concepts or reasons in logic and the underlying mathematics that clearly imply the existence and truth of axioms [5].

The other type of justification is *extrinsic justification*, which is proof that the axiom's results are consistent with the results produced by the rest of the axioms, and that the axiom's results can be proved by the other axioms alone (even if it may be a more complex proof) [5]. Furthermore, extrinsic justifications do not need their intrinsic counterparts to qualify an axiom.

3 Intrinsic Motivation for Including the Axiom of Choice

Due to the nature of axioms, some of our justifications must be pulled from concepts of what seems "natural" and logical, otherwise known as intrinsic motivations. In this section, I outline some of these arguments. I use the word "natural" to refer to a concept that are normal to us in the real world - such as selecting a physical object from a basket of objects.

The Idea of Choice

One of the most famous thought experiments relating to the Axiom of Choice is one given by Bertrand Russell. He asked that if you had an infinite number of pairs of socks - with the left and right socks of each pair clearly distinguishable - could you find a choice function? The answer is simple: you can always just pick the right sock of each pair. However, if each sock in each pair are indistinguishable, then how can we guarantee a well-defined choice function?

The Axiom of Choice allows us to assume that there is one. We cannot write down all infinite choices of socks we pick explicitly. Instead, the Axiom of Choice states that there are functions which can make those unlimited arbitrary choices for us per pair of socks. It seems intuitive that the socks should still be able to chosen from - that for each pair of socks, we can pick an arbitrary one to place into our new set. The original set has not intrinsically changed, apart from the loss of right and left labels (and even those are artificial - not necessarily inherent in socks or shoes). For a finite set of these pairs of socks, we can randomly pick from pairs to create this set regardless of the Axiom of Choice. Simply adding pairs to the original set until there are an infinite amount shouldn't change our approach. If we can do it for finite sets, then we should intuitively be able to continue to do it for each successive pair through a function.

Furthermore, mathematicians of all branches use this idea of choice without concern. It is central to fields like analysis, linear algebra, abstract algebra, measure theory, and general topology. Many proofs involve arbitrary choice in order to prove general statements and theorems (that are self-evident and intrinsic as well). In these fields, mathematicians tend to accept ZFC because of the positive consequences it has. It should be noted that the acceptance and positive usage of ZFC in other branches of mathematics seems to lend credibility to the Axiom of Choice.

The Iterative Conception of Sets

In the iterative conception of sets, all sets originate from a single set of objects, where these other sets are some selection of the elements of the original set. A common way of selecting these objects (and the way most mathematicians use the iterative conception) is by taking the power set, which is the set of all sets of elements from the original set, of the previous stage's set. This is also called the maximal iterative conception of sets, because it includes all subsets possible to create from the last stage's set. This is the conception I will focus on.

Because of the nature of such a hierarchy, for a set S_1 that shows up at stage s_1, the previous set S_0 at stage s_0 is the choice set for S_1. As the power set of S_0, S_1 has all the possible subsets of the elements in S_0, so for many of the non-joint combinations of these subsets, their choice set is just S_0. Denote the power set of S_1 as S_2 at stage s_2. The choice set of some combination of the elements in S_2 is simply S_1.

This has also been described as a "combinatorial understanding of the set-formation process" [7]. Taking the power set repeatedly allows us to ensure that every combination of elements exists somewhere in the set, so all of the choice sets exist. Accepting this combinatorial view of sets allows us to intrinsically justify the Axiom of Choice, because all choice sets naturally exist for any set of elements.

4 Extrinsic Motivation for Including the Axiom of Choice

If I introduced axioms into a consistent system that then became inconsistent, then one natural response would be to discard the added axiom because of its consequences. This is the basis for extrinsic motivation. Just as an axiom that disrupted a seemingly-good system would be discarded, an axiom that contributes to the seemingly-good results of a system should be included, especially if those results would be lost without the axiom. Zermelo agreed that although seemingly subjective, the self-evidence of axioms must contribute to the validity of mathematical principles [2]. In this section, I define some of the important and unique consequences of the Axiom of Choice and explain their significance to mathematics, extrinsically justifying why the Axiom of Choice's inclusion is beneficial.

Finite and Infinite Sets

A set is *Dedekind-infinite* if one can impose a bijection between the elements of some subset of the set to the elements of the set. Otherwise, this set is considered *Dedekind-finite*. The significance of this definition is that it was the first definition that does not rely on the natural numbers to prove that a set is infinite or finite.

This definition of infinite sets also has intrinsic motivations. No finite set has a bijection to any of its proper subsets - there will always be at least one fewer elements in the subset than the set. Infinite sets have subsets that are also infinite and are sometimes even in bijective correspondence to the set. Take, for example, the set of natural numbers $\mathbb{N}$. The function $\phi(n) = 2n$ from $\mathbb{N}$ to the subset of $\mathbb{N}$ containing the even natural numbers is bijective, so, in one sense, there are just as many even numbers as there are numbers. Another bijective function from $\mathbb{N}$ to a subset of $\mathbb{N}$ is $\phi(n) = n + 1$.

The Axiom of Choice is necessary here to prove that a set is Dedekind-finite in the sense that it has a finite number of elements. Without the Axiom of Choice, there is a model using only ZF that includes an infinite Dedekind-finite set. Here, the Axiom of Choice strengthens ZF to exclude seemingly contradictory results such as this one.

The intuitiveness of Dedekind's definition and the proof behind it helps indirectly support the Axiom of Choice. Thus, the model of set theory with the Axiom of Choice serves to stop contradictions like infinite Dedekind-finite sets from taking place.

The Well-Ordering Principle

Another extrinsic motivation for the Axiom of Choice comes from its relation to the Well-Ordering Principle in set theory. The idea of *ordering* a collection of objects is quite a natural one. Perhaps one may want to order them by color, shape, or size. In this section, I discuss the Well-Ordering Principle: its intrinsic motivations, its extrinsic motivations, and its relation to the Axiom of Choice.

The Well-Ordering Principle states that every set can be well-ordered. A set is *well-ordered* by a *strict total order* when each non-empty subset of that set has a least element (i.e., a minimal element). A strict total order refers to one where all pairs of elements are comparable and ordered against each other. It is established that the Axiom of Choice is equivalent to the Well-Ordering Principle and either one, if assumed true, can be used to prove the other. Well-ordering sets requires an infinite number of arbitrary decisions in choosing the order, and thus invokes the Axiom of Choice. For this reason, the intuitiveness of well-ordering gives the Axiom of Choice integrity for use, and vice versa. In this section I will outline justifications for the Well-Ordering Principle and argue that the strength of the principle lends credibility to the Axiom of Choice.

This principle has extrinsic motivations. One theorem that relies on well-ordering is the Prime Factorization Theorem (which itself seems self-evident - no two numbers would use the same factorization because they would then be the same number). This theorem has powerful consequences itself, as Gődel's Incompleteness Theorem uses it. The positive, powerful results of the Prime Factorization Theorem lends credibility to the well-ordering theorem as reliable and usable.

The Well-Ordering Principle also has intrinsic motivations. It seems natural that when defining an order to a set, there will be a minimal element - ordered such that its simply the first of the set. Even if the order function is arbitrary and requires arbitrary choice, it is inevitable that the first element to be chosen in the order can simply be called the minimal one.

Thus, the intrinsic and extrinsic motivations of the Well-Ordering Principle not only lend support to itself, but also its equivalent the Axiom of Choice.

Other Fields of Mathematics

In addition, there are many intuitive and basic results in other mathematical fields that are either necessary to it or have success. I will list some of the most important and basic ones that rely on the Axiom of Choice to illustrate the broad applications of the Axiom of Choice:

Zorn's Lemma This lemma states that each totally ordered subset of any non-empty, partially ordered set has a maximal element. It is equivalent to the Axiom of Choice and the Well-Ordering Principle, and has important applications to linear algebra.

Bases of vector spaces The Axiom of Choice is essential in proving that every vector space has a basis, which is a fundamental assumption to linear algebra. This is, crucially, also equivalent to Zorn's Lemma because the maximal linearly independent set of vectors are the bases for the corresponding vector space.

Cartesian Product Principle The Cartesian product of a nonempty family of nonempty sets is nonempty. This principle has intrinsic motivations and is important to set theory.

In addition, there are a few results that have not been proven to exist in a system without the Axiom of Choice, but have been proven to be false in all known systems without the Axiom of Choice.

Partition Principle If there is a surjection from a set A to a set B, then there is an injection from B to A.

Weak Partition Principle Any partition of a set A cannot be strictly larger than A.

These last two have natural intrinsic justifications, and lend credit to the Axiom of Choice's acceptance and usage.

5 Paradoxical Results of the Axiom of Choice

One of the main arguments against using the Axiom of Choice centres around the "paradoxes" that arise from certain usages of the Axiom of Choice. In this section, I will discuss common issues that the Axiom of Choice seemingly creates.

Types of Paradoxes

To begin, I must explain the main types of paradoxes to allow us to explore how some of the "contradictions" the Axiom of Choice leads still have truth to them. There are two that I will focus on: *veridical paradoxes* and *antimonies*.

Veridical paradoxes are those that hold some kind of truth - even if it may not be intuitive. For example, Galileo's Paradox is a veridical paradox. It states that there are just as many square numbers as natural numbers. This is true because the function $\phi(x) = x^2$ from the set of natural numbers to the set of square numbers is a bijection. However, this contradicts our intuition since it does not align with our other way to compare the size of a subset: if the subset is proper, then it lacks elements of the set containing it and is in this sense smaller than it. For finite sets, these two ways to compare the size of sets are equivalent; for infinite sets, these ways diverge. (Galileo concluded that normal comparisons like more or less do not apply to infinite sets).

Antimonies, on the other hand, are paradoxes that have no solutions and are completely self-contradicting no matter what logical reasoning is used. For example, the Liar Paradox is one. "I am a liar" is false if the speaker is truthful (and therefore not a liar), and it is true if the speaker is lying (and is a liar). Both solutions lead to the same conclusion: the sentence has no truth values that satisfy it.

I will now explore the main paradox that is frequently brought up as an counter-intuitive consequence of the Axiom of Choice.

Banach-Tarski Paradox

The Banach-Tarski Paradox states that if you take any two non-empty, bounded subsets A and B of $\mathbb{R}^3$, then you can divide A into a finite number of subsets that can be moved by rigid transformations - translations and rotations only - to form B.

A more intuitive equivalent is that if you have a solid 3-dimensional sphere, then you can decompose it into a finite number of disjoint subsets of points, which can be put back together in a different way such that one ends with two copies of the original ball (with one playful exemplifications is that this paradox allows us to turn a baseball into a sphere as large as the sun). Reassembling the sphere only requires rigid transformations of the points.

Crucially, this theorem uses the Axiom of Choice. When the sphere is decomposed, first as a hollow sphere and then extended to the points in the interior, the Axiom of Choice is invoked to produce that decomposition.

Many physicists and applied mathematicians feel uncomfortable with the implications of the Banach-Tarski Paradox in the real world. Of course, it is impossible to physically turn a baseball into a ball the size of the sun. However, it is important to remember that the Paradox relies on cutting up the sphere into sets that each contain an infinite number of points - something we cannot do in the real world in regards to physical matter. The atoms in a baseball are numerous, yet still finite, and so are the particles that constitute those atoms, presumably. Another crucial detail is that the paradox does not require us to define a volume to every subset of $\mathbb{R}^3$, so the sets we construct have no volume to preserve [10]. If one wanted to conserve volume, they could instead work with locales instead of topological spaces [9]. In addition, the idea of measurement in the proof of the paradox is much different from measurement in reality. The proof requires cutting the spheres into shapes with infinite sides - which is impossible with the defined size of atoms and particles in reality.

Furthermore, there is evidence that the Banach-Tarski Paradox has real implications for our world. Several papers have been published suggesting that because particles colliding at high energies can become different particles and more in number than before the collision is linked to the Banach-Tarski Paradox [1].

Thus, it seems that the Banach-Tarski Paradox is a veridical paradox, with elements of truth regardless of its intuition.

6 Constructivism and Other Views of the Axiom of Choice

There are several ways in which to modify the Axiom of Choice.

Constructive Definition of the Axiom of Choice

Constructivists are mathematicians that believe that for any mathematical object, it is necessary to construct examples of it in order to prove it exists. It is not enough, as most mathematicians do, to show assuming a principle's negation leads to contradiction (otherwise known as the *law of the excluded middle*).

For constructivists, the issue with the Axiom of Choice is that it does not guarantee us a way to find a choice function for every set (for example, the non-empty subsets of complex numbers). According to the constructivist definition, "exists" must be interpreted the same way as "find" - which implies that the Axiom of Choice is inherently false [8]. This type of definition leads to the loss of many of the results outlined in Section 5. Most mathematicians, for that reason, choose not to use this definition.

The Axiom of Countable Choice

Another variant of the Axiom of Choice is the Axiom of Countable Choice. It states that every set of a countable number of subsets has a choice function. This is a weaker axiom than the Axiom of Choice, but still suffices to prove a few of the results I discussed earlier (such as all Dedekind-infinite sets are infinite and that the union of countably many countable sets is countable). However, it is not powerful enough to prove the Well-Ordering Principle, and thus other results that rest on it (such as the Prime Factorization Theorem), which are still fundamental results. Thus, the Axiom of Choice is still preferable to Countable Choice.

The Axiom of Dependent Choice

The Axiom of Dependent Choice is another weakened form of the Axiom of Choice. It states that one may make a countable number of consecutive choices. This axiom is stronger than the Axiom of Countable Choice and weaker than regular Axiom of Choice. Indeed, it is implied by normal the Axiom of Choice and implies the Axiom of Countable Choice but is not implied by the Axiom of Countable Choice [6]. It does support more results than the Axiom of Countable Choice but fails to allow results specific to the Axiom of Choice, such as the Banach-Tarski Paradox [6], which, as I examined earlier, is reasonable and may have consequences in our reality. Thus, I do not see the Axiom of Dependent Choice as preferable to the Axiom of Choice.

7 Conclusion

I have outlined why the Axiom of Choice should in my view be included with the Zermelo-Frankel axioms as a preferred system of axioms. I outlined the ideas of choice, the intrinsic and extrinsic justifications of the Axiom, as well as its far-reaching effects on other fields of maths that naturally accept it. I described the Banach-Tarski Paradox that arises in systems that use the Axiom of Choice. Finally, I outlined several common alternatives to the Axiom of Choice, and argued that the Axiom was preferable to these. Future results in set theory will more concretely determine the status of the Axiom of Choice. With the current results, I believe all the positive results and equivalences of the Axiom of Choice qualify it to be accepted freely with the Zermelo-Frankel Axioms as a foundation of mathematics.

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