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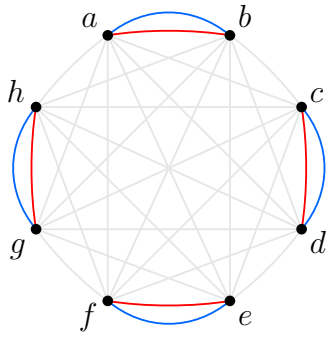
Problems 1691–1700

Q1691 Eight points a, b, c, d, e, f, g, h are drawn on a page. Four pairs are joined by red and blue curves, as shown in the diagram. All other pairs are joined by one line (or curve) which is shown as grey, and will be coloured either red or blue.



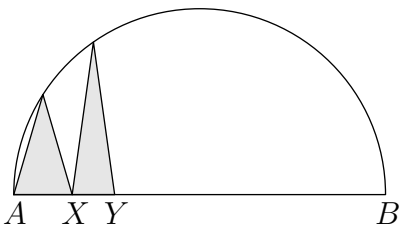
Prove that no matter how this colouring is done, the resulting diagram will contain three of the eight original points mutually joined by red lines, or four points mutually joined by blue lines.

Q1692 Prove that the sum of two different powers of 2 can never be a cube or higher power of an integer. That is, there are no solutions to the equation

$$2^a + 2^b = m^p$$

in which a, b, m, p are non–negative integers, $a \neq b$ and $p \geq 3$.

Q1693 In a semicircle on diameter AB we have $AX = 3$ and $XY = 2$. As shown in the diagram, two isosceles triangles of equal area have AX and XY as their bases, and their third vertices are on the semicircle. Find the diameter of the semicircle.



Q1694 Let a, m and n be positive integers, where m is odd. Find the greatest common factor of $a^m - 1$ and $a^n + 1$.

Q1695 Three groups of $2, 3$ and 4 passengers arrive independently and at random times at a railway station where a train departs every 12 minutes. What is the probability that the average waiting time per person is more than 8 minutes?

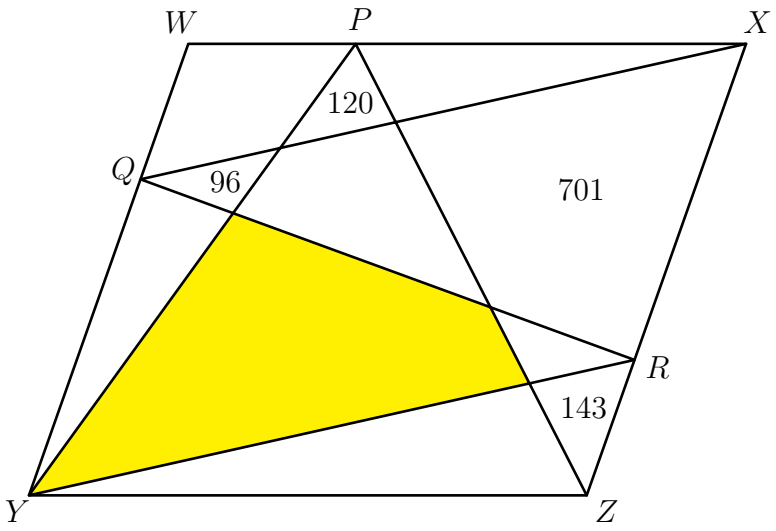
Q1696 Let n be an integer, $n \geq 2$, and let $p(x)$ be a polynomial with degree at most n, having integer coefficients. Suppose that the values of $p(x)$, where x is an integer, include all of the numbers $0, 1, 2, \ldots, n$. Prove that $p(x) = x + c$ for some constant c.

Q1697

(a) Each cell in a 3×4 rectangle has a value of 0 or 1. Prove that it is impossible for every row and column to have an odd sum. How many such rectangles can be constructed such that each row and column has an even sum?

(b) Each cell in $3 \times 4 \times 5$ box is assigned an integer value from 0 to 9. How many such boxes can be formed so that each line in all three directions has sum divisible by 10?

Q1698 In the diagram, $WXYZ$ is a parallelogram, and the numbers indicate the areas of certain subregions. Find the area of the coloured region.



Q1699 Suppose that

$$\left(\sqrt{20} + \sqrt{23}\right)^{2023} = a\sqrt{20} + b\sqrt{23}$$

where a and b are integers. Find the remainders when a and b are divided by 33.

Q1700 Find the sum of all natural numbers from 1 to 100 which have no common factor with 2022. Also, write the product of these numbers as an expression in terms of powers and factorials.