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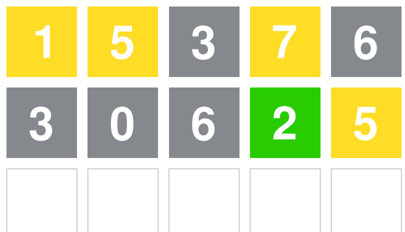
Solutions 1681–1690

Q1681 The recently popular game “Wordle” challenges you to guess a secret five–letter word. You may enter any word from the official “Wordle” word list, and you will be given some information in return. You are allowed a maximum of six guesses.

In the not–at–all well–known game “Squardle”, you have to guess a secret square number, and you may enter any five–digit square. Here is an example of the start of a game.



Two guesses have been entered so far: $124^2 = 15376$ and $175^2 = 30625$. When a digit is highlighted green in the diagram, it indicates that the digit occurs in the secret square in the same location as it is in the guess; a yellow highlight indicates a digit which occurs in the secret square, but not in the same location as in the guess; and a grey highlight indicates a digit which does not occur in the secret square at all. The secret square may contain the same digit more than once.

In Squardle, only three attempts are allowed. Can you win the game which was started above?

SOLUTION Let the digits of the secret square be $abcde$. We see immediately from the diagram that

$$d = 2\,, \quad a \neq 1\,, \quad b \neq 5\,, \quad e \neq 5\,.$$

The digits a, b, c, d, e consist of $1, 2, 5, 7$ and one digit x which we don’t yet know (and which may be a repeat of $1, 2, 5$ or 7). We also know from the diagram that $x \neq 0, 3, 6$, as these digits do not occur in the answer.

We begin by noting that the sum of the digits in a square is a multiple of 9 plus the sum of the digits in one of the numbers $0^2, 1^2, 2^2, \ldots, 8^2$: these digit–sums are $0, 1, 4, 7$ (with some repetitions). So $15 + x$ must be one of these plus a multiple of 9; this gives $x = 3$, which we have already ruled out, or $x = 1, 4, 7$, which remain as possibilities.

Next, the last digit of a square must be the last digit of one of the numbers

$$0^2, 1^2, 2^2, \ldots, 9^2 \;:$$

these digits are $0, 1, 4, 5, 6, 9$. So we can rule out $e = 7$, and we have $e = 1$ or $e = 4$, the latter only possible if $x = 4$.

Thirdly, the alternating sum $a - b + c - d + e$ is the remainder when the number is divided by 11, and for a square this must be the same as the remainder for one of the numbers $0^2, 1^2, 2^2, \ldots, 10^2$: that is, $0, 1, 3, 4, 5, 9$. The alternating sum can also be written as

$$a + b + c + d + e - 2b - 2d = 11 + x - 2b\,.$$

Now suppose that $x = 7$. Then we have $e = 1$, the only possibility for b is 7, and our options are 57721 and 75721. But by direct calculation, these are not squares.

Suppose that $x = 4$. Then $b = 1, 4$ or 7. The alternating sum is correspondingly $15 - 2b = 13, 7, 1$, and only the last is possible. So our options are 47521, 57124, 57421, and none is a square.

We are left with $x = 1$, so $b = 1, 7$ and the alternating sum $12 - 2b = 10, -2$ shows that the former is not possible. The latter *is* possible since it is 9 plus a multiple of 11, and it gives the only possibility for the secret square as $57121 = 239^2$.

Comment. If you are familiar with *modular arithmetic*, then you will be able to simplify some of these arguments. A nice introduction to modular arithmetic is given in an article by Bora Demirtas in this issue.

Q1682 Use the Arithmetic–Logarithmic–Geometric Mean Inequality (see the article by Toyesh Prakash Sharma in the last issue of *Parabola*) to prove (without a calculator!) that

$$e^{2/\sqrt{5}} < \frac{\sqrt{5}+1}{\sqrt{5}-1} < e\,.$$

SOLUTION As shown in the article, if x, y are positive numbers, then

$$\sqrt{xy} \;\le\; \frac{x-y}{\ln x - \ln y} \;\le\; \frac{x+y}{2}\;;$$

all three terms are positive, so we can take reciprocals and change the direction of the inequalities to give

$$\frac{2}{x+y} \;\le\; \frac{\ln x - \ln y}{x-y} \;\le\; \frac{1}{\sqrt{xy}}\,.$$

Now take

$$x = \frac{\sqrt{5}+1}{2} \quad \text{and} \quad y = \frac{\sqrt{5}-1}{2}\,.$$

Then we easily find

$$xy = 1\,, \quad x - y = 1\,, \quad x + y = \sqrt{5}\,,$$

and by using logarithmic laws we have

$$\ln x - \ln y = \ln\left(\frac{x}{y}\right) = \ln\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right);$$

therefore, the inequality becomes

$$\frac{2}{\sqrt{5}} \;\le\; \ln\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right) \;\le\; 1\,.$$

Taking the exponential of all terms (which is valid since the exponential is an increasing function) yields the result claimed.

Q1683 For background on this problem, see the article by Timothy Hume in the previous issue of *Parabola*. If A and B are points on a sphere, then we shall write $\widehat{AB}$ for the distance on the sphere between A and B. We denote the radius of the sphere by R.

(a) Use the coordinate formulae (2) and (3) in the article to prove the arc–length formula (8).

(b) Prove that if the great circle arcs AC and BC intersect at right angles, then

$$\cos\frac{\widehat{AB}}{R} = \left(\cos\frac{\widehat{AC}}{R}\right)\left(\cos\frac{\widehat{BC}}{R}\right).$$

(c) (For readers who have studied advanced calculus.) What do you get from the result of (b) if the arcs AB, BC and AC are very small compared with R?

SOLUTION Let A and B be the points with coordinates

$$\begin{aligned}(x_1,y_1,z_1) &= (R\sin\lambda_1\cos\phi_1,\ R\cos\lambda_1\cos\phi_1,\ R\sin\phi_1)\\ (x_2,y_2,z_2) &= (R\sin\lambda_2\cos\phi_2,\ R\cos\lambda_2\cos\phi_2,\ R\sin\phi_2)\end{aligned}$$

as in Timothy Hume's article. Then the straight–line distance ("tunnel distance") between A and B is given by

$$\begin{aligned}D^2 &= (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\\ &= R^2(\sin\lambda_1\cos\phi_1-\sin\lambda_2\cos\phi_2)^2\\ &\quad+R^2(\cos\lambda_1\cos\phi_1-\cos\lambda_2\cos\phi_2)^2\\ &\quad+R^2(\sin\phi_1-\sin\phi_2)^2\,.\end{aligned}$$

This looks pretty nasty, but when we expand, we'll get terms like

$$\sin^2\lambda_1\cos^2\phi_1+\cos^2\lambda_1\cos^2\phi_1+\sin^2\phi_1=\cos^2\phi_1+\sin^2\phi_1=1\,.$$

Doing this very carefully gives

$$D^2=R^2\big(2-2\sin\phi_1\sin\phi_2-2\cos\phi_1\cos\phi_2\cos\lambda_1\cos\lambda_2-2\cos\phi_1\cos\phi_2\sin\lambda_1\sin\lambda_2\big)\,,$$

and the $\cos(x-y)$ formula simplifies this to

$$D^2=R^2\big(2-2\sin\phi_1\sin\phi_2-2\cos\phi_1\cos\phi_2\cos(\lambda_1-\lambda_2)\big)\,.$$

Now draw a diagram. (We leave this up to you!) You will see that the angle θ subtended at the centre of the sphere by the arc AB satisfies

$$\sin\frac{\theta}{2} = \frac{\frac{1}{2}D}{R} = \frac{D}{2R}$$

and hence

$$\cos\theta = 1 - 2\sin^2\Big(\frac{\theta}{2}\Big) = 1 - \frac{D^2}{2R^2} = \sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2\cos(\lambda_1-\lambda_2)\,.$$

Therefore, the arc length is given by

$$\widehat{AB} = R\theta = R\arccos\big(\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2\cos(\lambda_1-\lambda_2)\big)\,,$$

and this proves formula (8).

Comment. It's not easy to get complicated algebra right first time, and it's therefore always worth doing some common–sense checks. This formula should give the distance from pole to pole when $\phi_1 = 90°$ and $\phi_2 = -90°$: we get $\widehat{AB} = R\arccos(-1) = \pi R$, which is clearly correct. We might also note that if the difference between the longitudes λ_1 and λ_2 gets smaller (leaving the latitudes unchanged), then $\widehat{AB}$ also becomes smaller, which makes sense.

The simplest way to approach part (b) is to realise that we can rotate the whole sphere, leaving distances unaltered, so that C becomes the north pole. Then C has latitude $90°$. If A is now at latitude ϕ_1 and longitude λ_1, and B is at ϕ_2, λ_2, then

$$\angle AOC = 90° - \phi_1\,,\quad \cos\frac{\widehat{AC}}{R} = \cos(90° - \phi_1) = \sin\phi_1$$

and similarly

$$\cos\frac{\widehat{BC}}{R} = \sin\phi_2\,.$$

Since the arcs AC and BC intersect at right angles, $\lambda_1 - \lambda_2 = \pm 90°$, and so the formula from (a) gives

$$\cos\frac{\widehat{AB}}{R} = \sin\phi_1\sin\phi_2 = \left(\cos\frac{\widehat{AC}}{R}\right)\left(\cos\frac{\widehat{BC}}{R}\right)$$

as claimed. Readers who have studied Maclaurin series in calculus will know that there is a very good approximation

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

if θ is small. So if the arcs in (b) are small compared with the radius we have

$$1 - \frac{\widehat{AB}^2}{2R^2} \approx \Big(1 - \frac{\widehat{AC}^2}{2R^2}\Big)\Big(1 - \frac{\widehat{BC}^2}{2R^2}\Big)\,,$$

which can be simplified to

$$\widehat{AB}^2 \approx \widehat{AC}^2 + \widehat{BC}^2 - \frac{\widehat{AC}^2\,\widehat{BC}^2}{2R^2}\,.$$

If AC and BC are small compared with R, then the last term on the right hand side is going to be very very small, and we have the approximate relation

$$\widehat{AB}^2 \approx \widehat{AC}^2 + \widehat{BC}^2$$

when AC and BC are perpendicular. That is, Pythagoras' Theorem holds approximately on a small part of a sphere. Since it is very hard to distinguish a small part of a sphere from a flat surface, this makes perfect sense!

Q1684

(a) Let $p(x) = 1 + 2x + 3x^2 + 4x^3$. Find a polynomial $q(x)$ with integer coefficients, not all zero, such that when $p(x)q(x)$ is expanded and terms collected, there will be no terms in x^k unless k is a square number. (Note that 0 is a square: so we want a product polynomial that looks like $a + bx + cx^4 + dx^9 + \cdots$.)

(b) Prove that if we replace $p(x)$ by any polynomial with integer coefficients, a polynomial $q(x)$ with this property can always be found.

SOLUTION We shall consider a product

$$(1 + 2x + 3x^2 + 4x^3)(\cdots) = \cdots\,,$$

and shall build up the second factor on the left hand side one term at a time in such a way as to achieve what we want. Start with

$$\begin{aligned} p(x) &= 1 + 2x + 3x^2 + 4x^3 \\ q(x) &= a + bx \\ p(x)q(x) &= a + (2a+b)x + (3a+2b)x^2 + (4a+3b)x^3 + 4bx^4\,. \end{aligned}$$

Now, we don't care what coefficients we get for the constant and the x term, but we want the x^2 coefficient to be zero. So we cancel out the $(3a+2b)x^2$ in the product by adding an opposite term to $q(x)$: our next attempt is

$$\begin{aligned} p(x) &= 1 + 2x + 3x^2 + 4x^3 \\ q(x) &= a + bx - (3a+2b)x^2 \\ p(x)q(x) &= a + (2a+b)x - (2a+b)x^3 - (9a+2b)x^4 - (12a+8b)x^5\,. \end{aligned}$$

Now this will have only square exponents, provided that

$$2a + b = 0\,,\quad 12a + 8b = 0\,.$$

Unfortunately, as we have here two linear equations in two unknowns, we expect to get only the trivial solution $a = b = 0$, which does not solve our problem. (In some cases we might be lucky and get a non–trivial solution – check for yourself that in this case we don't.) Therefore we continue modifying $q(x)$ so as to eliminate the x^3 term:

$$\begin{aligned} p(x) &= 1 + 2x + 3x^2 + 4x^3 \\ q(x) &= a + bx - (3a + 2b)x^2 + (2a + b)x^3 \\ p(x)q(x) &= a + (2a + b)x - 5ax^4 - (6a + 5b)x^5 + (8a + 4b)x^6 \,. \end{aligned}$$

At this stage we do not need to eliminate the x^4 term, so we take any x^4 term in $q(x)$ and then continue to eliminate unwanted terms,

$$\begin{aligned} p(x) &= 1 + 2x + 3x^2 + 4x^3 \\ q(x) &= a + bx - (3a + 2b)x^2 + (2a + b)x^3 + cx^4 \\ p(x)q(x) &= a + (2a + b)x - (5a - c)x^4 - (6a + 5b - 2c)x^5 + (8a + 4b + 3c)x^6 + 4cx^7 \end{aligned}$$

and

$$\begin{aligned} p(x) &= 1 + 2x + 3x^2 + 4x^3 \\ q(x) &= a + bx - (3a + 2b)x^2 + (2a + b)x^3 + cx^4 + (6a + 5b - 2c)x^5 \\ p(x)q(x) &= a + (2a + b)x - (5a - c)x^4 + (20a + 14b - c)x^6 \\ &\quad + (18a + 15b - 2c)x^7 + (24a + 20b - 8c)x^8 \,. \end{aligned}$$

At this stage, to eliminate all terms with non–square exponent, we'll need to solve three equations in three unknowns, which, as mentioned above, is unlikely to be successful. But if we continue for just one more step. . .

$$\begin{aligned} p(x) &= 1 + 2x + 3x^2 + 4x^3 \\ q(x) &= a + bx - (3a + 2b)x^2 + (2a + b)x^3 + cx^4 + (6a + 5b - 2c)x^5 - (20a + 14b - c)x^6 \\ p(x)q(x) &= a + (2a + b)x - (5a - c)x^4 - (22a + 13b)x^7 \\ &\quad - (36a + 22b + 5c)x^8 - (80a + 56b - 4c)x^9 \end{aligned}$$

. . . we now have only two equations in three unknowns (because we don't care what coefficient we have for x^9), and a non–zero solution is guaranteed. The equations are

$$22a + 13b = 0 \,, \quad 36a + 22b + 5c = 0 \,,$$

and with careful working it is not difficult to find an integer solution $a = 65$, $b = -110$, $c = 16$ (among other possibilities). Checking, we can substitute these values back into the last equation to yield

$$\begin{aligned} p(x) &= 1 + 2x + 3x^2 + 4x^3 \\ q(x) &= 65 - 110x + 25x^2 + 20x^3 + 16x^4 - 192x^5 + 256x^6 \\ p(x)q(x) &= 65 + 20x - 309x^4 + 1024x^9 \,, \end{aligned}$$

and, as required, all exponents in this product are squares.

For part (b) we take any polynomial

$$p(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$$

with integer coefficients, and we seek to show that there is a $q(x)$ with integer coefficients such that the product $p(x)q(x)$ has only terms with square exponents. Inspired by the answer to (a), we take a polynomial with unknown coefficients and degree $n^2 - n$, say

$$q(x) = q_0 + q_1 x + q_2 x^2 + \cdots + q_{n^2-n} x^{n^2-n} .$$

We begin by observing that this polynomial has $n^2 - n + 1$ coefficients, presently unknown. Now the product will have terms up to x^{n^2}, so $n^2 + 1$ terms in all. This will include a constant term and terms in x, x^4, $x^9, \ldots, x^{n^2}$, and there are $n + 1$ of these. We don't care what coefficients are attached to these terms, but we want all the *other* terms, $n^2 - n$ of them, to have coefficient zero. To achieve this, we need to solve the equations

$$\begin{aligned}
p_2q_0 + p_1q_1 + p_0q_2 &= 0 \\
p_3q_0 + p_2q_1 + p_1q_2 + p_0q_3 &= 0 \\
p_5q_0 + p_4q_1 + p_3q_2 + p_2q_3 + p_1q_4 + p_0q_5 &= 0 \\
&\vdots \\
p_nq_{n^2-n-1} + p_{n-1}q_{n^2-n} &= 0 .
\end{aligned}$$

In these equations, the p_k are the known coefficients, which are integers, the q_k are the unknowns, and the left hand sides are the coefficients of x^2, x^3, $x^5, \ldots, x^{n^2-1}$ in the product $p(x)q(x)$. Since the system involves $n^2 - n$ equations in $n^2 - n + 1$ unknowns, there is certain to be a solution with integer values for $q_0, q_1, q_2, \ldots, q_{n^2-n}$, and this solves the problem.

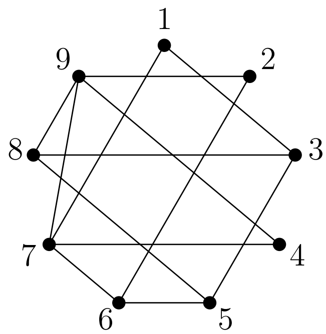
Comment. There is actually nothing special here about the squares, except that there are infinitely many of them. For example, we could give an almost identical proof to show that for any $p(x)$ there exists $q(x)$ such that the exponents in the product are only primes, or only Fibonacci numbers, or… or only any infinite set of non–negative integers.

Q1685

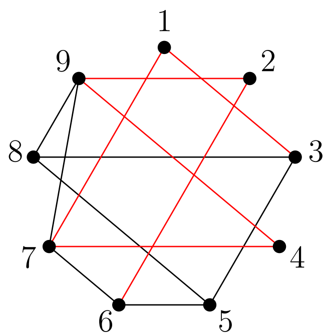
(a) Show how to arrange the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ around a circle in such a way that the sum of two neighbouring numbers is never a multiple of 3 or 5 or 7. In how many ways can this be done?

(b) Given any 9 consecutive integers, is the same task always possible?

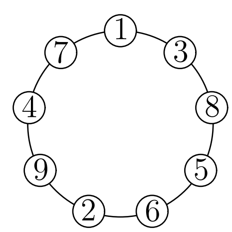
SOLUTION For the first problem, draw a diagram consisting of nine points labelled $1, 2, 3, 4, 5, 6, 7, 8, 9$, and a line joining two points whenever their sum is "permissible", that is, not a multiple of 3 or 5 or 7.



We wish to select nine of the lines in such a way as to connect up all nine points and return to the first point. Since $1, 2$ and 4 only have two lines available, the lines shown in red below are obligatory:



It is now easy to see that the only way to complete a circuit is to use the lines 3–8–5–6. So there is only one solution to the problem, if rotations and reflections are counted as the same solution.



For question (b), consider nine consecutive integers and let k be the middle number, so that the least is $k - 4$ and the greatest is $k + 4$. Sums of two of these will be somewhere near $2k$. To find a value of k which is most likely to fail, we look for a set of consecutive numbers in which as many as possible are multiples of 3 or 5 or 7. This can be done systematically (look up the *Chinese Remainder Theorem* if you would like to pursue this), but trial and error is probably just as easy; we find that the numbers $48, 49, 50, 51$ and $54, 55, 56, 57$ all have forbidden factors and are thus "impermissible" sums. So choosing $2k = 52$, that is, $k = 26$, means that the sums of k with the other eight numbers are $48, 49, 50, 51, 53, 54, 55, 56$. As only one of these sums is allowable, 26 can only have one number adjacent to it, and no circle can be formed with the numbers $22, 23, 24, 25, 26, 27, 28, 29, 30$.

Comment. As the problem depends only on divisibility by $3, 5, 7$, the existence (or not) of a solution with middle number k is effectively the same as the existence for $k+105$. Another impossible case occurs when $2k = 53 + 105$, that is, $k = 79$. These are not the only cases: an exhaustive computer search reveals that the arrangement is impossible for $k = 25, 26, 27, 28, 77, 78, 79, 80$ and possible for all other k from 1 to 105. In some of these cases, every number has at least two possible neighbours, but the neighbours cannot "fit together" in a manner which completes a full circle.

Q1686 Simplify

$$\frac{\sqrt[3]{560 + 158\sqrt{2} + 324\sqrt{3} + 90\sqrt{6}}}{\sqrt[3]{560 - 158\sqrt{2} + 324\sqrt{3} - 90\sqrt{6}}}\,.$$

SOLUTION First we try to simplify the cube roots, beginning with the numerator. By careful but straightforward algebra, we have

$$\big(a+b\sqrt{2}+c\sqrt{3}\big)^3 = a(a^2+6b^2+9c^2)+b(3a^2+2b^2+9c^2)\sqrt{2}+c(3a^2+6b^2+3c^2)\sqrt{3}+(6abc)\sqrt{6}\,,$$

and so we would like to find a, b, c such that

$$\begin{aligned} a(\ \ a^2 + 6b^2 + 9c^2) &= 560\,,\\ b(3a^2 + 2b^2 + 9c^2) &= 158\,,\\ c(3a^2 + 6b^2 + 3c^2) &= 324\,,\\ 6abc &= 90\,. \end{aligned}$$

Now a system of equations like this is generally going to be very hard to solve, but perhaps we can guess an answer? The last equation gives $abc = 15$: let's guess that a, b, c are all positive integers. In the first equation, $a^2 + 6b^2 + 9c^2$ is then a positive integer, so a is a factor of 560; but 3 is not a factor of 560 and so 3 is not a factor of a. Similarly, the second equation shows that 3 is not a factor of b and 5 is not a factor of b; the third shows that 5 is not a factor of c. Since $abc = 15$, the only remaining possibility is $a = 5$, $b = 1$, $c = 3$. Now it is important to substitute back and check this, because the whole procedure depends on a, b, c being integers, which was a guess, not a certainty. However, this is a matter of simple arithmetic, and we leave it to you to confirm that our guess is correct, and hence that

$$\big(5 + \sqrt{2} + 3\sqrt{3}\big)^3 = 560 + 158\sqrt{2} + 324\sqrt{3} + 90\sqrt{6}\,.$$

To obtain the denominator, all we need do is change the sign of b: thus

$$\big(5 - \sqrt{2} + 3\sqrt{3}\big)^3 = 560 - 158\sqrt{2} + 324\sqrt{3} - 90\sqrt{6}\,.$$

Therefore, the expression is

$$\frac{5 + \sqrt{2} + 3\sqrt{3}}{5 - \sqrt{2} + 3\sqrt{3}} = \frac{\big(5 + \sqrt{2} + 3\sqrt{3}\big)\big(5 - \sqrt{2} - 3\sqrt{3}\big)}{\big(5 - \sqrt{2} + 3\sqrt{3}\big)\big(5 - \sqrt{2} - 3\sqrt{3}\big)} = \frac{\sqrt{2} + 3\sqrt{3}}{5}\,.$$

Q1687

(a) A line with gradient m intersects the ellipse $x^2 + 2y^2 = 3$ at the point $(1,1)$ and another point. Find the other point.

(b) Find all triples of positive integers a, b, c having no common factor such that $a^2 + 2b^2 = 3c^2$.

SOLUTION Substituting $y = m(x-1)+1$ into the equation of the ellipse and collecting terms gives the quadratic equation

$$(2m^2+1)x^2 - 4m(m-1)x + (2m^2 - 4m - 1) = 0\,.$$

Instead of solving by the quadratic formula, note that the two roots must add up to $4m(m-1)/(2m^2+1)$. Since we know that one of the solutions is $x = 1$, we can easily find the other, and then substitute back into the equation of the line to find the corresponding y value. After simplification – please check the algebra yourself – we obtain

$$x = \frac{2m^2-4m-1}{2m^2+1}\,, \quad y = \frac{-2m^2-2m+1}{2m^2+1}\,, \tag{$*$}$$

which is the other point of intersection between the line and the ellipse.

For part (b), we first note that if m is rational, then the values of x, y in (a) are also rational. Moreover, the formulae give, with one exception, *all possible* rational points (x, y) satisfying the equation $x^2 + 2y^2 = 3$. For if (x, y) is a pair of rational numbers, then the line between (x, y) and $(1, 1)$ has gradient

$$m = \frac{y-1}{x-1}\,,$$

which is rational, and so (x, y) is given by the above procedure with this value of m. There are two points where this argument does not quite work.

- If (x, y) is the same as $(1, 1)$, then we have only one point on the line and m is not defined. However, it is easy to check that we do in fact get $(x, y) = (1, 1)$ by taking $m = -\frac{1}{2}$, which is the gradient of the tangent to the ellipse at $(1, 1)$.
- If $(x, y) = (1, -1)$, then the line joining the points is vertical and does not have a (finite) gradient. If you take $x = 1$, $y = -1$ in equations $(*)$ and try to solve for m, then you will find that there is no solution. This is the exception referred to above – the only rational point on the ellipse which is not given by the formulae $(*)$.

Since m is rational, we can write $m = u/v$, where u, v are integers. Making this substitution and multiplying numerators and denominators by v^2 gives

$$x = \frac{2u^2-4uv-v^2}{2u^2+v^2}\,, \quad y = \frac{-2u^2-2uv+v^2}{2u^2+v^2}\,.$$

Notice that if $u = 1$, $v = 0$, then $x = 1$, $y = -1$, so we have regained the missing point mentioned above. To sum up: by taking u and v to be integers, these latest formulae give all rational solutions of the equation $x^2 + 2y^2 = 3$.

Now let a, b, c be integers satisfying $a^2 + 2b^2 = 3c^2$. By excluding the trivial solution $a = b = c = 0$, we can divide by c^2 to obtain

$$\left(\frac{a}{c}\right)^2 + 2\left(\frac{b}{c}\right)^2 = 3$$

– in essence, exactly the equation we have just studied! Since a/c and b/c are rational, they are given by

$$\frac{a}{c} = \frac{2u^2 - 4uv - v^2}{2u^2 + v^2}, \quad \frac{b}{c} = \frac{-2u^2 - 2uv + v^2}{2u^2 + v^2},$$

where u, v are integers. It may be that we can cancel a common factor out of both fractions (for example, if v is even then both numerators and denominators have a factor of 2); therefore, all possible integer solutions of $a^2 + 2b^2 = 3c^2$ having no common factors are given by taking

$$a = 2u^2 - 4uv - v^2, \quad b = -2u^2 - 2uv + v^2, \quad c = 2u^2 + v^2$$

with u, v integers, and then cancelling any common factors. The solution $a = b = c = 0$, which we ignored previously, is now given by the values $u = v = 0$.

Q1688 A one–person game is played as follows. Begin with a stack of n coins. Split them into two (non–empty) stacks with say a and b coins; this move gives a score of ab. Keep splitting the remaining stacks until all stacks consist of a single coin, and add all the scores. For example, starting with a stack of 30 coins, we might split it into stacks of 20 and 10, scoring 200; then into 20 and 7 and 3 scoring 21, total score so far 221; and so on until we have 30 stacks each containing one coin.

Prove that, no matter how the coins are split, the final total score will always be the same.

SOLUTION We can illustrate the game on a diagram by drawing a point for each coin, and a line between two points whenever the coins are in the same stack. We begin with n points, and a line between each pair of points; so there are $\frac{1}{2}n(n-1)$ lines. Splitting the original stack (over the course of one or more moves) into a number of smaller stacks will separate the points into smaller groups, in each of which every pair of points has a line joining them. Now splitting any stack, whether it is the original or a smaller stack, into groups of size a and b, means deleting all the lines between these groups. There are ab such lines, and this is the score accumulated for the move. Finishing the game means deleting all the lines in our original diagram, and the total final score will therefore be the number of initial lines, that is, $\frac{1}{2}n(n-1)$.

Q1689 An ant walking across the floor noticed a grain of ant poison and a grain of sugar. Hating poison and loving sugar, the ant decided to walk in such a way that its distance from the poison increases at the same rate at which its distance to the sugar decreases. The ant was surprised to discover that no matter how fast it walked, it could not reach the sugar this way.

(a) Explain why the ant could not reach the sugar as long as it moved in the way described.

(b) Are there any exceptional cases when the ant could reach the sugar?

(c) Describe the path of the ant in case (a).

SOLUTION When two quantities change at equal and opposite rates, their sum remains constant. Therefore as long as the ant moves in the way described, the sum of its distances to the two grains will equal $AP + AS$, where AP is the distance from the ant to the poison when the motion first begins, and AS is the distance from the ant to the sugar when the motion first begins. Now imagine that the ant reaches the sugar. In that case, the sum of its distances to the two grains would equal PS, the distance between the grains. Thus we would have $PS = AP + AS$. But, unless the ant started exactly on a straight line between P and S, this would violate the triangle inequality for triangle APS. Therefore, the ant cannot reach the sugar as long as it walks in the way described.

...and this argument also shows that the only case in which the ant might reach the sugar is when it begins its journey directly between the two grains: then it will head straight towards the sugar and will eventually reach it... as long as it walks fast enough and for long enough.

The path of the ant in (a) will be a curve such that at every point on the curve, the sum of the distances to the two grains will be constant. Therefore the ant will travel along an arc of an ellipse that has the two grains as foci.

Q1690 In how many different ways can 10^{100} (a googol) be factorised as xyz, where x, y, z are positive integers,

(a) if the order of the factors does matter, for example, $2^{50} \times 5^{50} \times 10^{50}$ is regarded as different from $5^{50} \times 10^{50} \times 2^{50}$?

(b) if the order of the factors does not matter, for example, $2^{50} \times 5^{50} \times 10^{50}$ is the same as $5^{50} \times 10^{50} \times 2^{50}$?

SOLUTION We can write the factorisation as

$$10^{100} = (2^{a_1}5^{b_1}) \times (2^{a_2}5^{b_2}) \times (2^{a_3}5^{b_3}) ,$$

where $a_1, \ldots, b_3$ are non–negative integers satisfying

$$a_1 + a_2 + a_3 = 100 , \qquad b_1 + b_2 + b_3 = 100 .$$

For part (a) of the question, there are no further restrictions on the exponents. We can specify a choice for the a_k which are the exponents of 2 by saying "two" one hundred times and "next" twice. For example,

"two, two, next, two, two, two, next, two, two, …, two"

would correspond to $a_1 = 2$, $a_2 = 3$, $a_3 = 95$. T he number of ways of making this choice is the number of ways of choosing 2 locations out of 102 for the words "next", and this is $C(102, 2) = 5151$. Exactly the same reasoning applies to the b_k, so the total number of choices, and the total number of ways to write 10^{100} as a product of three positive integers, is $5151^2 = 26532801$.

When order matters, we have shown in (a) that there are 5151^2 possible factorisations. To tackle the case in which order does not matter, we need to consider when a factorisation can contain two or more equal terms. First, 10^{100} is not a cube; so there is no factorisation of the form xxx. To count factorisations in which two of the factors are the same and the other is different,

- choose whether the first, second or third factor is to be the different one: there are 3 options;
- choose the repeated factor: it must be a number $x = 2^a 5^b$ such that $2^{2a}5^{2b}$ is a factor of 10^{100}, so $0 \le a, b \le 50$: there are 51^2 options.

So there are altogether 3×51^2 factorisations involving a repeated factor, leaving $5151^2 - 3 \times 51^2$ with no repeated factor. As we are now considering the case where order *does not* matter, the former consist of the expressions we want, counted three times each (since xxz and xzx and zxx are all regarded as the same); the latter are counted six times each: so the number of different factorisations is

$$\frac{3 \times 51^2}{3} + \frac{5151^2 - 3 \times 51^2}{6} = 4423434\,.$$