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An elementary geometric proof that regular polygons have the largest area

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In [1], R. Tanaka and others give an interesting proof of the following statement.

Theorem. *A regular polygon has the largest area among all polygons inscribed in a circle.*

In this note, we present a simpler proof which requires only elementary geometry. The idea of the proof in [1] is to take the average of two angles; that is, to perform the operation $(\alpha, \beta) \to (\frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2})$. By comparison, the idea of our proof is to make one of two angles the desired value; that is, the operation $(\alpha, \beta) \to (\frac{360^\circ}{n}, \alpha + \beta - \frac{360^\circ}{n})$. This idea allows the operations to end in a finite number of steps, and simplifies the proof.

From here on, we present our proof.

Lemma. *Consider two points A and B on a circle centred at a point O. Take two points C and D on the arc AB satisfying $\angle AOC < \angle AOD < \angle COB$. Then the area of $\triangle ADB$ is larger than that of $\triangle ACB$.*

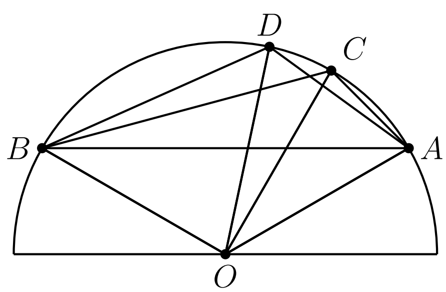

Figure 1

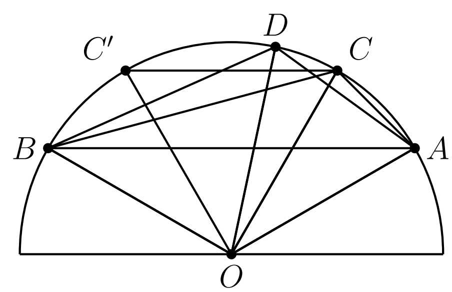

Figure 2

Proof. Take a point C' on the arc AB satisfying $\angle C'OB = \angle AOC$. Then the line CC' is parallel to AB. Since $\angle AOC < \angle AOD$, the point D lies on the arc CB. Since $\angle AOD < \angle COB$ and $\angle AOC = \angle C'OB$, we have that

$$\angle DOB = \angle AOC + \angle COB - \angle AOD > \angle C'OB.$$

So the point D lies on the arc AC'. Therefore, D lies on the arc CC'. Since D is higher than C from the line AB, the area of $\triangle ADB$ is larger than that of $\triangle ACB$. □

Proof of Theorem. Consider any non-regular n-gon inscribed in a circle centred at a point O. Draw rays from O to all vertices of the n-gon, and divide it into n isosceles triangles. Even if these isosceles triangles are rearranged, they form an n-gon of the same area. So we need not care about the order of the isosceles triangles.

Consider the following operation for such an n-gon.

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Operation. Rearrange n isosceles triangles so that the smallest and largest inner angles are neighbours. Denote the smallest by $\triangle OP_1P_2$ and the largest by $\triangle OP_2P_3$. Take a point P_2' on the arc P_1P_3 satisfying $\angle P_1OP_2' = \frac{360^\circ}{n}$. Replace $\triangle P_1P_2P_3$ by $\triangle P_1P_2'P_3$. Then we have a new n-gon.

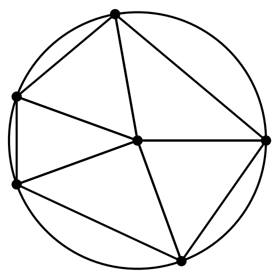

Figure 3 : Original n-gon

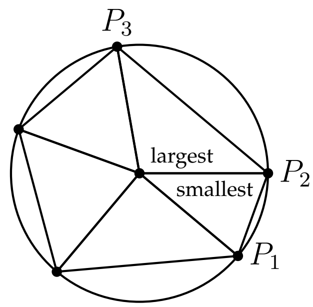

Figure 4 : Rearrange isosceles triangles

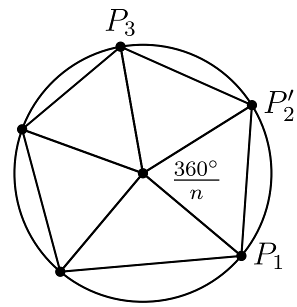

Figure 5 : Replace $\triangle P_1P_2P_3$ by $\triangle P_1P_2'P_3$

The average of the central angles is equal to $\frac{360^\circ}{n}$. Since the average is between the smallest and the largest, we have that $\angle P_1OP_2 < \angle P_1OP_2' < \angle P_2OP_3$. By the lemma, the area of $\triangle P_1P_2'P_3$ is larger than that of $\triangle P_1P_2P_3$. This means that the area of the new n-gon is larger than that of the former one.

The operation makes one of central angles equal to $\frac{360^\circ}{n}$, and increases the area. If we repeat the operation sufficiently many times (at most $n-1$), then all central angles become $\frac{360^\circ}{n}$. Therefore, a finite sequence of operations makes the n-gon regular and increases its area. Hence, the area of the regular n-gon is larger than that of the original n-gon. □

References

[1] R. Tanaka, J. Miyamoto, Y. Maruo, K. Nakayama and R. Miyadera, An elementary proof that the regular polygon is the largest among polygons that are inscribed in a circle, *Parabola* **59** (2023), 1–8.