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Goats, partitions, triangles and necklaces

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1 Introduction

The goat problem [1] is as follows. There is a herd of n goats which have the odd habit of dividing into new groups every so often. These groups are formed according to

Rule R:

> One goat from each group leaves, to together form a new group,
> while all other goats remain in their respective groups.

This process occurs repeatedly throughout the day until at some point the number of groups remains constant. If there are 7 groups at this point, then what must n be?

Solution: The number of groups will only remain constant at 7 when the group sizes are $M = (1, 2, 3, 4, 5, 6, 7)$, because R applied to M produces M, i.e., $R(M) = M$. Since the elements of M sum to 28, the answer to the goat problem is $n = 28$.

Remark 1. It may take a long time to reach this steady-state. For example, from an initial grouping of four equal groups of size 7, it requires 38 applications of the rule R to produce the partition M.

This problem suggests a generalisation: given an arbitrary initial grouping of n goats, which values of n will produce a constant number of groups with repeated applications of R? A solution to this problem will be formed later in this paper.

2 Partitions and the operator H

The goat problem will be put aside for now but two aspects of it shall be retained, namely the rule R and partitions. Instead of R, the operator H is introduced - which does the same thing but applies to partitions of integers instead of to groups of goats.

Definition 2 (Partition)**.** A *partition* P of an integer n is a multiset of positive integers whose sum is n. $|P|$ is the number of elements in P. [3]

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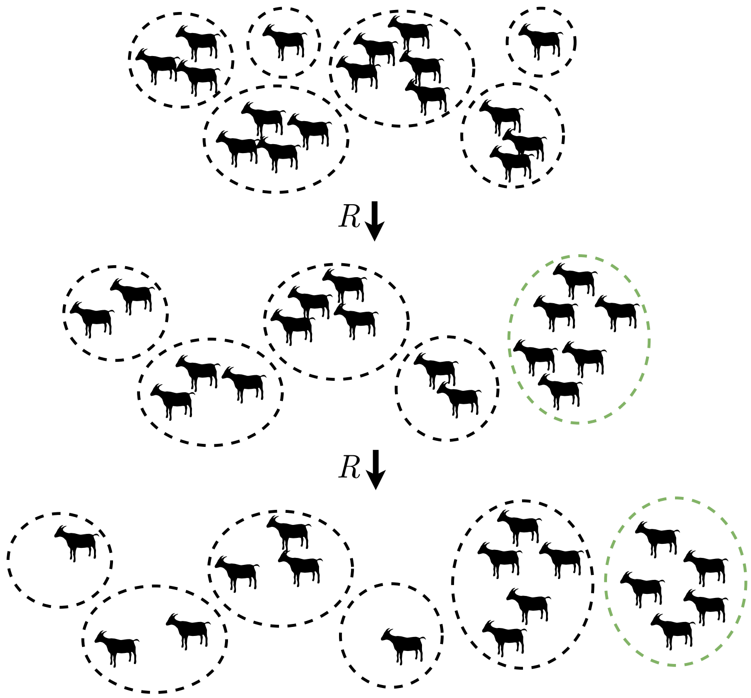

Figure 1: The goat division rule R applied to an initial division of goats. [2]

In this paper, the elements of a partition will be written in ascending order, enclosed in parentheses. For example $(1, 1, 4, 5)$ is one of many partitions of 11, and $|P| = 4$. Figure 2 represents this partition visually as rows of boxes - these are called *Young diagrams* or *Ferrers diagrams*. The number of boxes in each row represents the size of each element within the partition.

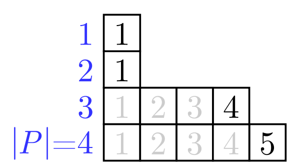

Figure 2: Partition $P = (1, 1, 4, 5)$.

The larger n is, the more partitions there are. The longest of these is $(1, 1, \ldots, 1)$ with n elements, and the shortest is (n) with only one element.

Definition 3 (The operator H)**.** The operation H on a partition P reduces each element of P by one, and adds the new element $|P|$. All zero elements are discarded. The result of applying H to P is $H(P)$. This is a well-defined operation on the set of all partitions.

Definition 4 (Powers of H)**.** The k-th power of H is the operation of applying k times the operation H. Therefore, $H^0(P) = P$ and $H^k(P) = H(H^{k-1}(P))$ for all $k \geq 1$.

Example 5. *If $P = (1, 1, 1, 3, 6)$, then*

$$\begin{aligned} H(P) &= (2, 5, 5) \\ H^2(P) &= H(H(P)) = (1, 3, 4, 4) \\ H^3(P) &= H(H^2(P)) = (2, 3, 3, 4) \end{aligned}$$

...and so on. These examples are shown in detail in Figure 3, in which each new element of $H^k(P)$ is indicated in grey. Note that $|H(P)|$ can be smaller, the same or larger than $|P|$.

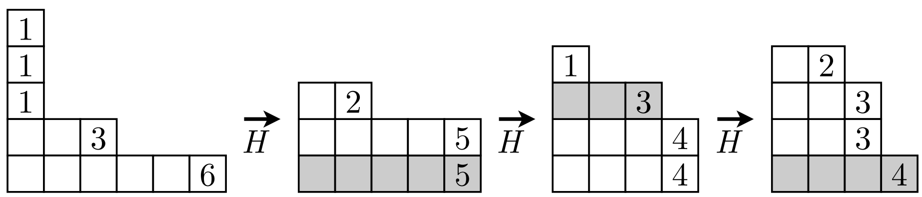

Figure 3: $H(P)$, $H^2(P)$ and $H^3(P)$ for $P = (1, 1, 1, 3, 6)$.

Theorem 6. *If P is a partition of n, then $H^k(P)$ is a partition of n.*

Proof. H reduces each of the $|P|$ elements of P by 1 and adds the element $|P|$ to $H(P)$. The sum of elements in P and $H(P)$ are the same, so P and $H(P)$ are partitions of the same number. Therefore, this is true of any number of applications of H to P. □

Theorem 7. *If P contains exactly k 1's, then $|H(P)| = |P| - k + 1$.*

Proof. If P contains k 1's, then they are all eliminated in $H(P)$, the other $|P| - k$ elements in $H(P)$ remain greater than 0, and the new element $|P|$ is added to $H(P)$. Thus, $|H(P)| = |P| - k + 1$. □

Corollary 8.
If P contains no 1's, then $|H(P)| = |P| + 1$.
If P contains one 1, then $|H(P)| = |P|$.
If P contains more than one 1, then $|H(P)| < |P|$.

Example 9. *The following partitions illustrate the statements of Corollary 8.*

$$\begin{aligned} P &= (2, 3) & H(P) &= (1, 2, 2) \\ P &= (1, 2, 2) & H(P) &= (1, 1, 3) \\ P &= (1, 1, 3) & H(P) &= (2, 3) \end{aligned}$$

3 Cycles and standard partitions

Definition 10 (The operator H^*)**.** $H^*(P)$ is the shortest sequence of partitions

$$P, H(P), \ldots, H^k(P)$$

that contains $H^{i+1}(P)$ for each $i \geq 0$.

Example 11. $H^*(1,3,5) = (1,3,5), (2,3,4), (1,2,3,3), (1,2,2,4), (1,1,3,4)$ *since the next partition in the sequence is* $(2,3,4)$ *which is already in the sequence - see Figure 4.*

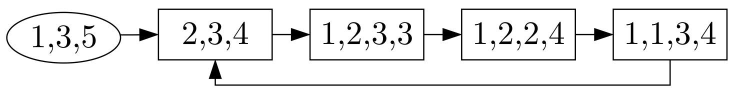

Figure 4: The sequence $H^*(1,3,5)$.

Definition 12 (Cycles)**.** A *cycle* is a sequence $H^*(P)$ whose final element Q satisfies $H(Q) = P$. Put more simply, $H^*(P)$ is a cycle if it recurs.

Example 13. *In Figure 4,* $H^*(1,3,5)$ *is not a cycle but contains the cycle* $H^*(2,3,4)$*.*

Definition 14 (Equivalent cycles)**.**
Two cycles are *equivalent* if they contain the same partitions.

Example 15. $H^*(2,3,4)$*,* $H^*(1,2,3,3)$ *and* $H^*(1,1,3,4)$ *are equivalent - see Figure 4.*

Theorem 16. *The set of partitions of* n *contains one or more cycle.*

Proof. Since there are finitely many partitions of n, $H^*(P)$ must be of finite length for any partition P of n. Therefore the final element Q of $H^*(P)$ starts a cycle, since $H(Q)$ is a element of $H^*(P)$. □

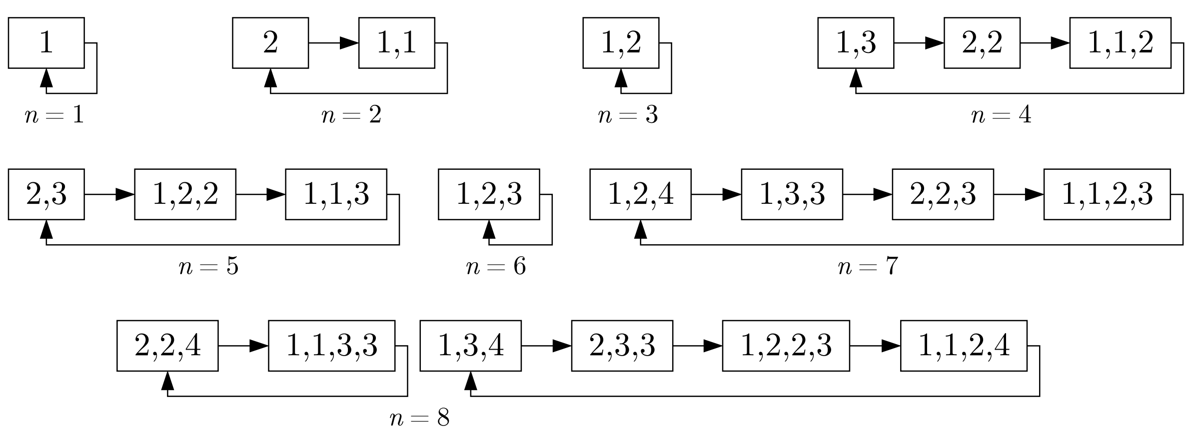

Figure 5: Cycles for $n = 1$ to 8.

The cycles for $n = 1$ to 8 are listed in Figure 5. Note that $n = 8$ is the first n for which there are has two cycles.

Definition 17 (Triangle numbers)**.** The *triangle number* T_m is the sum $1 + 2 + \cdots + m$ - see Figure 6. It is not hard to show that $T_m = m(m + 1)/2$.

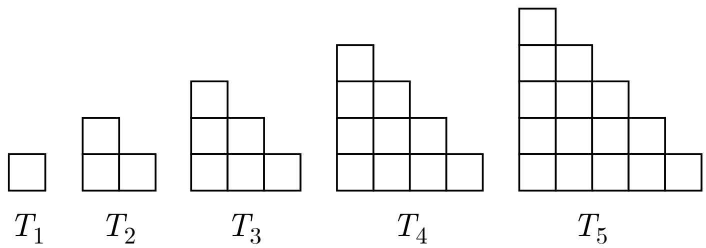

Figure 6: Triangle numbers.

Note that in Figure 5, if n is a triangle number - i.e., $1, 3, 6$ - then its corresponding cycle has length one [4].

Definition 18 (Standard partitions)**.** A *standard partition* of n, where $n = T_m + r < T_{m+1}$, is one that can be constructed from the sequence $0, 1, \ldots, m$ by adding 1 to r of the elements.

Example 19. *$n = 12$ satisfies $n = T_4 + 2 = 10 + 2$. Some standard partitions of 12 can be constructed as follows:*

$$(0, 1, 2, 3, 4) + (1, 1, 0, 0, 0) = (1, 2, 2, 3, 4)$$
$$(0, 1, 2, 3, 4) + (1, 0, 1, 0, 0) = (1, 1, 3, 3, 4)$$
$$(0, 1, 2, 3, 4) + (0, 1, 0, 1, 0) = (0, 2, 2, 4, 4)$$
$$(0, 1, 2, 3, 4) + (0, 0, 0, 1, 1) = (0, 1, 2, 4, 5)$$

These can be formed using the T_4 base - see Figure 7 in which T_4 is indicated with white boxes.

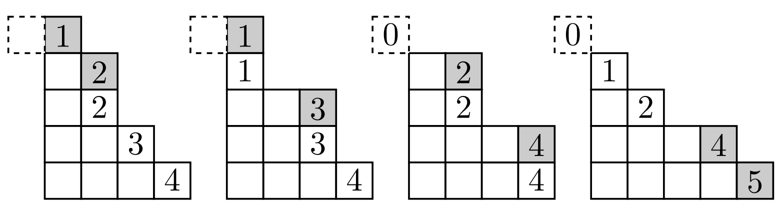

Figure 7: Standard partitions $(1, 2, 2, 3, 4)$, $(1, 1, 3, 3, 4)$, $(0, 2, 2, 4, 4)$ and $(0, 1, 2, 4, 5)$.

Definition 18 entails using 0 in some standard partitions. Therefore, standard partitions P may be represented based on $n = T_m + r < T_{m+1}$ as $(p_0, p_1, \ldots, p_m)$, where p_0 may be 0 or 1.

Definition 20 (The partition $P - M$)**.** If P is standard for $n = T_m + r < T_{m+1}$ and $M = (0, 1, \ldots, m)$, then $P - M$ consists of the elements $x_j = p_j - j$ for $j = 0, \ldots, m$. Note that r of the x_j are 1 and $m + 1 - r$ are 0 since $p_j = j + 1$ for r j's, and $p_j = 0$ for $m + 1 - r$ j's.

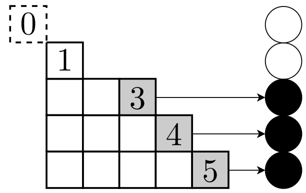

Figure 8: $P - M = (0, 0, 1, 1, 1)$ for $P = (0, 1, 3, 4, 5)$, indicated by black and white beads.

Example 21. *$P = (0, 1, 3, 4, 5)$ is standard for $n = 13 = T_4 + 3$. Since $M = (0, 1, 2, 3, 4)$, $P - M = (0, 0, 1, 1, 1)$ - see Figure 8.*

Theorem 22. *Let $n = T_m + r < T_{m+1}$ and $P - M = (p_0 - 0, p_1 - 1, \ldots, p_m - m)$ be $(x_0, x_1, \ldots, x_m)$. Then $H(P) - M = (x_1, x_2, \ldots, x_m, x_0)$ and, more generally for all $k \leq m$,*

$$H^k(P) - M = (x_k, x_{k+1}, \ldots, x_m, x_0, x_1, \ldots, x_{k-1}) .$$

Proof. We have $P = p_0, p_1, \ldots, p_m$. Hence,

$$P - M = (p_0, p_1 - 1, p_2 - 1, \ldots, p_m) - m = (x_0, x_1, \ldots, x_m).$$

Moreover,

$$H(P) = (h_0, h_1, \ldots, h_m) = (p_1 - 1, p_2 - 1, \ldots, p_{m-1} - 1, m + p_0).$$

We deduce that

$$\begin{aligned} H(P) - M &= (h_0, h_1 - 1, h_2 - 2, \ldots, h_m - m) \\ &= (p_1 - 1, p_2 - 2, p_3 - 3, \ldots, p_0) \\ &= (x_1, x_2, x_3, \ldots, x_m, x_0) \end{aligned}$$

The result for $H^k(P)$ follows from repetitions of the result for $H(P)$. □

A partition P may be determined to be standard as follows: find n by adding the elements of P. Then $n = T_m + r = m(m+1)/2 + r$ gives m and r. If $|P|$ is not m or $m + 1$, then P is not standard. Then compare P with $M = (0, 1, \ldots, m)$, adding a 0 to P if necessary to make P and M of the same length. Then subtract M from P. This gives $P - M$ in which there are r 1's, and the rest 0's.

Example 23. *For $P = (1, 3, 3, 5, 6)$, $n = 18 = T_5 + 3$, so $m = 5$, $r = 3$ and $|P| = 5 = m$. Add a 0 to equate the lengths of P and $M = (0, 1, 2, 3, 4, 5)$; that is, $P = (0, 1, 3, 3, 5, 6)$. Then $P - M = (0, 0, 1, 0, 1, 1)$, so P is standard.*

Theorem 24. *Suppose that $n = T_m + r < T_{m+1}$. If $r = 0$, then there is only one standard partition P of n and $|P| = m$. If $r > 0$, then the standard partitions P of n have $|P| = m$ or $m + 1$ elements, ignoring 0.*

Proof. This follows directly from Definition 18. □

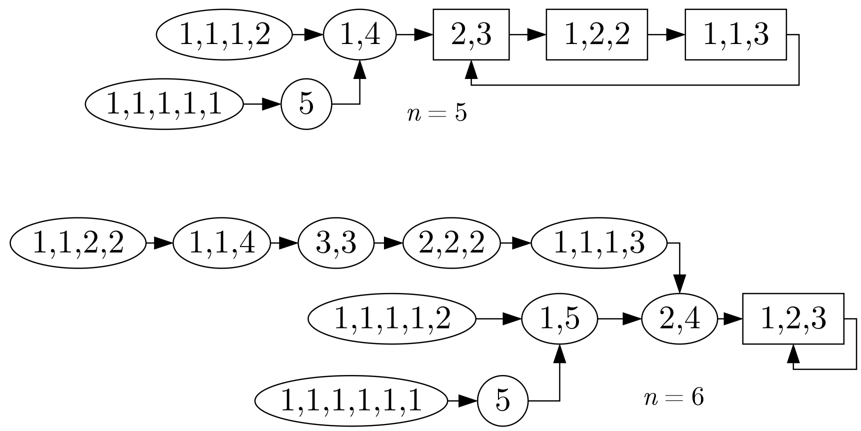

Figure 9: The partitions for $n = 5$ and $n = 6$ and their connections via H.

Example 25. *Figure 9 shows all possible partitions of $n = 5$ and $n = 6$ and how they are connected via the operator H. For $n = 5 = T_2 + 2$, there are three standard partitions of length $|P| = 2$ or 3. For $n = 6 = T_3$, the only standard partition of n is $(1, 2, 3)$, which can also be written as $(0, 1, 2, 3)$.*

Definition 26 ($C(n, r)$)**.**
$C(n, r) = n!/(r!(n - r)!$ is the number of r-sized subsets of a set of n objects.

Example 27. *There are $C(8, 3) = 8!/(3! \times 5!) = 56$ possible sets of size 3 drawn from a set of 8 distinct objects.*

Theorem 28. *If $n = T_m + r < T_{m+1}$, then there are $C(m + 1, r)$ standard partitions of n.*

Proof. Let $P = p_0, p_1, \ldots, p_m$ be a standard partition for n. There are r values of j that satisfy $p_j = j + 1$. There are $m+1$ positions, including p_0, and thus $C(m+1, r)$ standard partitions of n. □

Example 29. *If $n = T_2 + 2 = 5$, then n has $C(3, 2) = 3$ standard partitions - see Figure 9. If $n = T_5 + 4 = 19$, then n has $C(6, 4) = 15$ standard partitions, of which one is*

$$(0, 1, 2, 3, 4, 5) + (0, 1, 0, 1, 1, 1) = (0, 2, 2, 4, 5, 6) \,.$$

There are 14 others.

Table 1 compares the number of standard partitions with the total number of partitions. Note that $C(m + 1, r)$ is not monotonic as a function of n, since it depends on both m and r and is a small fraction of the total number of partitions as n increases.

Theorem 30. *If P is standard for $n = T_m + r < T_{m+1}$, then $|H(P)| = m$ or $m + 1$.*

Proof. Since P is standard, $|P| = m$ or $m + 1$.
If P has no 1, then $|P| = m$ and $|H(P)| = |P| + 1 = m + 1$.
If P has one 1, then $|P| = |H(P)| = m$ or $m + 1$.
If P has two 1's, then $|P| = m + 1$ and $|H(P)| = |P| - 1 = m$.
A standard partition P must have 0, 1 or 2 1's, so $H(P) = m$ or $m + 1$. □

n	m	r	Standard Partitions	All Partitions [3]
5	2	2	3	7
10	4	0	1	42
20	5	5	6	627
30	7	2	28	5604
40	8	4	126	37338

Table 1: Number of standard and all partitions for different values of n.

Theorem 31. *Let h_j be the elements of $H(P)$ for any standard partition $P = p_0, p_1, \ldots, p_m$. Then $h_j = p_{j+1} - 1$ for each $j = 0, 1, \ldots, m-1$ and $h_m = m + p_0$.*

Proof. $H(P)$ is the non-descending sequence of $m+1$ numbers $p_1 - 1, \ldots, p_m - 1, |P|$, which are labelled h_0 to h_m. Thus, $h_j = p_{j+1} - 1$ for each $j = 0, 1, \ldots, m-1$, and $h_m = |P| = p_0 + m$. □

Theorem 32. *If P is a standard partition, then $H^*(P)$ are all standard partitions.*

Proof. It is sufficient to show that if P is standard, then $H(P)$ is standard.

Suppose that $n = T_m + r < T_{m+1}$ If $r = 0$, then $n = T_m$ and the only standard partition of n is $P = (0, 1, \ldots, m)$. Since $H(P) = P$ in this case, $H(P)$ is standard if P is standard. If $r > 0$, then the elements of a standard partition $P = p_0, p_1, \ldots p_m$ satisfy $p_j = j$ or $j+1$ for $j = 0, 1, \ldots, m$ by Theorem 31.

If P is standard, then $|H(P)| = m$ or $m+1$, by Theorem 30. By Theorem 31, the elements of $H(P)$ are $h_j = p_{j+1} - 1$ for $j = 0, 1, \ldots, m-1$, and $h_m = m + p_0$. Since p_{j+1} is $j+1$ or $j+2$ for $j = 0, 1, \ldots, m-1$, h_j is j or $j+1$ for $j = 0, 1, \ldots, m-1$, and h_m is m or $m+1$. Therefore, $H(P)$ is standard. □

4 Inverses of P

Definition 33 (Inverse of a partition)**.**
An *inverse* of a partition P is any partition Q satisfying $H(Q) = P$.

For a given integer n, there are many partitions. Each partition P has a unique sequence $H(P)$ but not every partition P has an inverse. For example $P = (1, 1, 2, 2)$ has no inverse. Some partitions have multiple inverses: for instance $(1, 2, 3)$ has itself and $(2, 4)$ as inverses - see Figure 9. A formula may be found for the number of inverses of a partition P, based on how many 1's and elements larger than 1 it has.

Theorem 34. *Let $P = (1^k, x_1, x_2, \ldots, x_r)$ be any partition of n, where 1^k means k 1's and $1 < x_j \leq x_{j+1}$ for all j. Then P has an inverse Q if and only if $x_j \geq |P| - 1$ for some x_j in P.*

Proof. Suppose that $P = (1^k, x_1, x_2, \ldots, x_r)$ where $1 < x_1 \leq x_2 \leq \cdots \leq x_r$ and that P has at least one inverse Q. Then 1^k in P comes from 2^k in Q and x_i in P comes from

$x_i + 1$ in Q except if x_j is $|Q|$ for some x_j. Suppose that Q has t 1's. Then Q has the form $(1^t, 2^k, x_1 + 1, x_2 + 1, \ldots, x_r + 1)$ less some $x_j + 1$.

Both P and Q represent the same n. Therefore, $n = k + \sum x_i$, some $x_j = |Q|$, and $n = t + 2k + \sum(x_i + 1) - (x_j + 1) = t + 2k + \sum x_i + r - (x_j + 1)$. It follows that $k + \sum x_i = t + 2k + \sum x_i + r - (x_j + 1)$. Thus, $x_j + 1 = k + t + r$, so $t = x_j + 1 - k - r$ for some j. Since $k + r = |P|$, this simplifies to $t = x_j + 1 - |P|$. Therefore, $t \geq 0$ if and only if $x_j \geq |P| - 1$. □

Example 35. *Let $P = (1, 2, 4)$. Then $x_j \geq |P| - 1 = 2$ has two solutions, $x_2 = 2$ and $x_3 = 4$, so P has two inverses, namely $(1, 1, 2, 3)$ and $(2, 5)$. Now let $P = (1, 1, 1, 4)$. Then $x_j \geq |P| - 1 = 3$ has only one solution, namely $x_4 = 4$, so P has a unique inverse, namely $Q = (1, 2, 2, 2)$. Finally, let $P = (1, 1, 1, 2, 3)$. Then $x_j \geq |P| - 1 = 4$ has no solution for x_j and so P has no inverse - see Figure 10.*

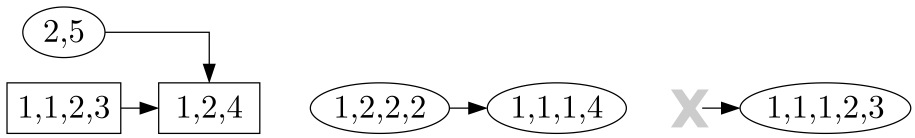

Figure 10: Inverses of $(1, 2, 4)$, $(1, 1, 1, 4)$ and $(1, 1, 1, 2, 3)$.

Theorem 36. *A standard partition has exactly one standard inverse.*

Proof. Theorem 31 states that $h_j = p_{j+1} - 1$ for $j = 0, 1, \ldots, m - 1$ and $h_m = p_0 + m$. A standard partition P determines $H(P)$, which is also standard, and P is also an inverse of $H(P)$. The equations above are reversed: $p_{j+1} = h_j + 1$ for $j = 0, 1, \ldots, m - 1$ and $p_0 = h_m - m$. Given $H(P)$, these determine P, so P is unique as a standard inverse of $H(P)$. Note that $H(P)$ may also have a non-standard inverse. □

Let us now look at the problem of finding an inverse. Given $P = 1^k, x_1, x_2, \ldots, x_r$, determine which x_j are $|P| - 1$ or greater. Then $t = x_j + 1 - |P|$ is the number of 1's in Q and k is the number of 2's in Q, where Q is an inverse of P. An inverse of P is therefore the partition Q with all $x_i + 1$ except for $x_j + 1$. s

Example 37. *Let us construct the inverse of $H(P) = (0, 1, 3, 4, 5)$. Here, $n = 13 = T_4 + 3$, so $m = 4$. Then $p_{j+1} = h_j + 1$ for $j = 0, 1, \ldots, m - 1$ and $p_0 = h_m - m$. Thus,*

$$
\begin{aligned}
p_0 &= h_4 - 4 = 5 - 4 = 1 \\
p_1 &= h_0 + 1 = 0 + 1 = 1 \\
p_2 &= h_1 + 1 = 1 + 1 = 2 \\
p_3 &= h_2 + 1 = 3 + 1 = 4 \\
p_4 &= h_3 + 1 = 4 + 1 = 5
\end{aligned}
$$

Hence, $P = (1, 1, 2, 4, 5)$, the standard inverse of $(0, 1, 3, 4, 5)$ - shown in Figure 11.

Example 38. *Let $P = (1, 2, 4)$; then $|P| - 1 = 2$.*
If $x_j = 2$, then $t = x_j + 1 - |P| = 3 - 3 = 0$ and $k = 1$, so $Q_1 = (2, 5)$.
If $x_j = 3$, then $t = 3 + 1 - 3 = 2$ and $k = 1$, so $Q_2 = (1, 1, 2, 3)$.

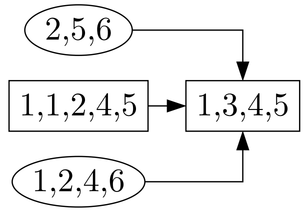

Figure 11: All inverses of $(1, 3, 4, 5)$.

Theorem 39. *The number of inverses of $P = (1^k, x_1, x_2, \ldots, x_r)$ is the number of distinct elements $x_j > |P| - 1$. If $x_r < |P| - 1$, where x_r is the largest element in P, then P has no inverse. Otherwise, P has one or more inverses.*

Proof. By Theorem 34, the number of inverses of P is the number of distinct elements x_j satisfying $x_j \geq |P| - 1$. Each of these gives a different value of $t = x_j + 1 - |P|$, the number of 1's in Q. □

Example 40. *If $P = (1, 1, 1, 2, 6)$, then $p_4 = 6 > |P| - 1 = 4$, so P has an inverse Q - but only one, since $p_3 = 2 < |P| - 1$. Now, $t = 6 + 1 - 5 = 2$, so $Q = (1, 1, 2, 2, 2, 3)$.*

Theorem 39 can be used to settle a possible issue: could a standard cycle lack an entry point from a non-standard partition, i.e., be isolated from the other standard partitions?

Theorem 41. *For $n > 1$, every standard cycle has an entry point from a non-standard partition. Equivalently, for $n > 1$ every cycle has a element which has more than one inverse.*

Proof. If $n = T_m + r < T_{m+1}$, then a standard partition P for n will have two inverses if and only if two values of p_j are greater than or equal to $|P| - 1$, by Theorem 39. Now, $|P|$ is m or $m + 1$. Two of the values p_j which could be greater than or equal to $m - 1$ or m are p_{m-1} and p_m. Since p_m is m or $m + 1$, $p_m > |P| - 1$. If $|P| = m$, then $p_{m-1} \geq |P| - 1$. By Theorem 39, if $H^*(P)$ is of length $m + 1$, then $m + 1 - r$ elements of $H^*(P)$ will have $|H^k(P)| = m$, so at least $m + 1 - r$ elements of $H^*(P)$ will have two inverses. Therefore, at least one of $H^*(P)$ will have two inverses, one of which is non-standard and serves as an entry point for the cycle $H^*(P)$.

Note that a partition may have more than two inverses, depending on the number of values of p_j that satisfy $p_j \geq |P| - 1$. For example, $(2, 3, 4, 5)$ has three inverses - shown in Figure 12.

If $H^*(P)$ has partitions of unequal length, then it will still have at least one $H^k(P)$ of length m. If $|H^*(P)|$ is $(m + 1)/k$, then k will divide both $m + 1 - r$ and r. Now, $H^*(P)$ has $(m + 1 - r)/k \geq 1$ elements of length m and so $H^*(P)$ will have an entry point in this case as well. □

Example 42. *If $P - M = (1, 1, 1, 0, 1)$, then*

$$H^*(P) - M = (1, 1, 1, 0, 1)\,,\ {}^*(1, 1, 0, 1, 1)\,,\ {}^*(1, 0, 1, 1, 1)\,,\ {}^*(0, 1, 1, 1, 1)\,,\ (1, 1, 1, 1, 0)$$

The three starred items - corresponding to the partitions

$$(1, 2, 2, 4, 5)\,,\ (1, 1, 3, 4, 5)\,,\ (0, 2, 3, 4, 5)$$

have two inverses each, shown in Figure 12.

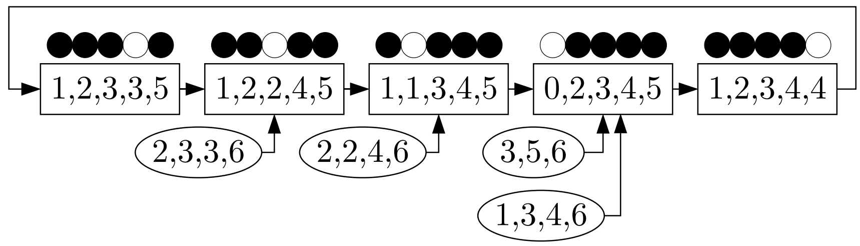

Figure 12: Cycle with multiple non-standard entry points.

Theorem 43. *The number of partitions for n having no inverse is greater than or equal to the number of cycles for n.*

Proof. Every $H^*(P)$ ends in a cycle. Thus, to each P there is a unique cycle. However, two or more partitions may have the same cycle. Every cycle lies in some $H^*(P)$ where P has no inverse. Thus, the number of cycles cannot exceed the number of partitions without inverse but may be smaller. □

Example 44. $n = 5$ *has the partitions*

$$(5)\,,\ (4,1)\,,\ (3,2)\,,\ (3,1,1)\,,\ (2,2,1)\,,\ (2,1,1,1)\,,\ (1,1,1,1,1)\,.$$

Of these, $(2,1,1,1)$ *and* $(1,1,1,1,1)$ *have no inverses. The only cycle is* $H^*(1,2,2)$.

Theorem 45. *If P is a standard partition, then $H^*(P)$ is a cycle.*

Proof. Now, $H^*(P) = P$, $H(P)$, …,$H^k(P)$ for some k, all of which are standard partitions. It needs to be established that $H^{k+1}(P) = P$. By the definition of $H^*(P)$, $H^{k+1}(P)$ is one of $P, H(P), \ldots, H^k(P)$. Suppose that $H^{k+1}(P) = H^m(P) = Q$ where $m > 0$, so that $H^*(Q)$ is a cycle but $H^*(P)$ is not. Then Q has two standard inverses, one being $H^{m-1}(P)$, since $H^m(P) = Q$, but also $H^k(P)$, since $H^{k+1}(P) = Q$. However, Theorem 36 shows that a standard partition can have only one standard inverse. Hence, $m = 0$ and $H^{k+1}(P) = P$, and $H^*(P)$ is a cycle. □

Remark 46. *The application of H reduces P to a circular rotation of the elements of $P - M$. After $m + 1$ rotations, $H^{m+1}(P) = P$, which is the usual cycle length. However, certain periodic configurations of $P - M$ will rotate to themselves in fewer rotations. An example occurs with the cycle* $H^*(0,2,2,4) = (0,2,2,4), (1,1,3,3)$. *Subtracting* $M = (0,1,2,3)$ *gives* $(0,1,0,1)$ *and* $(1,0,1,0)$ *which are periodic sequences and thus have a shorter cycle length - see Figure 13.*

Theorem 47. *If P is a standard partition for n and $T_m < n < T_{m+1}$, then the cycle $H^*(P)$ is of length $m+1$ or a divisor of $m+1$. If $|H^*(P)| = m+1$, then $H^*(P)$ contains r elements of length $m+1$, and $m+1-r$ elements of length m.*

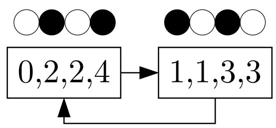

Figure 13: Cycle $H^*(0,2,2,4)$ is shorter as the necklace $P-M$ is periodic.

Proof. By Theorem 22, H rotates the 1's and 0's in $P-M$ to the left. If they are not periodic, then the rotation has $m+1$ steps. If the 1's are periodic, then the period length must divide $m+1$. The 0^{th} position will be visited once by each of the r 1's. Each time that happens, the partition length will be $m+1$. Therefore, there are r partitions of length $m+1$. □

Example 48. *Let $P=(2,3,4)$ be a standard partition of 9. Then $n=9$, $m=3$ and $r=3$, and $H^*(2,3,4)$ has $r=3$ partitions of length $m+1=4$ and $m+1-r=1$ of length $m=3$. Also, $H^*(2,3,4)=(2,3,4),(1,2,3,3),(1,2,2,4),(1,1,3,4)$ - see Figure 4.*

5 The number of cycles

Definition 49. A *necklace* is a circular sequence of elements. It can be rotated but not flipped over.

Theorem 50. *If $n=T_m+r$, then n has as many cycles as there are necklaces with $m+1-r$ white beads and r black beads.*

Proof. $P-M$ consists of $m+1-r$ 0's and r 1's in a certain arrangement depending on P. A cycle for P corresponds to a cycle for $P-M$. If the arrangement of 0's and 1's is not periodic, then $P-M$ and P will have a cycle of length $m+1$. If the arrangement of 0's and 1's is periodic, then $P-M$ will have a cycle of length $(m+1)/k$, where k is the number of periods. This is possible only when r and $m+1-r$ have common divisors. The arrangement of 0's and 1's correspond to a sequence of white and black beads, forming a necklace. The values of $H^k(P)$ will correspond to rotations of the necklace. A periodic necklace will have fewer distinct rotations. □

Example 51. *If $P=(0,2,3,4)$, then the corresponding necklace is given by $P-M=(0,1,1,1)$. This necklace is then rotated throughout the sequence $H^*(2,3,4)$ - see Figure 14.*

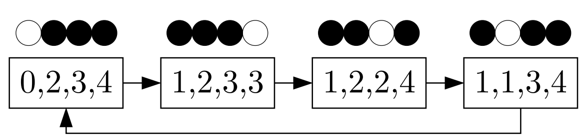

Figure 14: The sequence $H^*(2,3,4)$ with corresponding necklaces above each partition.

The following section indicates how to calculate the number of necklaces with x white and y black beads, based on an unpublished paper by the first author.

5.1 The number of black-and-white necklaces

Definition 52 ($S(x, y)$)**.**
$S(x, y) = C(x + y, x) = (x + y)!/(x!y!)$ is the number of sequences with x 0′s and y 1′s.

Example 53. $S(3, 5) = 8!/(3!5!) = 56.$

Definition 54 ($G(x, y)$)**.**
$G(x, y)$ is the number of non-periodic sequences with x 0′s and y 1′s. This number is defined recursively as follows: $G(x, y) = S(x, y) - \sum_r G(x/r, y/r)$ where each r divides both x and y and is greater than 1. Note that if no such r exists, then $G(x, y) = S(x, y)$.

Example 55.

$$
\begin{aligned}
G(2,3) &= S(2,3) = 10 \\
G(4,6) &= S(4,6) - G(2,3) = 210 - 10 = 200 \\
G(6,9) &= S(6,9) - G(2,3) = 5005 - 10 = 4995 \\
G(12,18) &= S(12,18) - \big(G(12/2,18/2) + G(12/3,18/3) + G(12/6,18/6)\big) \\
&= S(12,18) - \big(G(6,9) + G(4,6) + G(2,3)\big) \\
&= 86\,493\,225 - (4995 + 200 + 10) \\
&= 86\,488\,020\,.
\end{aligned}
$$

Theorem 56. *The number of necklaces having x white and y black beads is*

$$N(x, y) = \sum_r \frac{r\,G(x/r, y/r)}{x + y}\,.$$

where $r \geq 1$ divides x and y. The number of necklaces having r periods is $rG(x, y)/(x + y)$, where r divides both x and y. The case $r = 1$ corresponds to non-periodic necklaces.

This enumerates the non-periodic and periodic necklaces.

Example 57. *The number of necklaces with 12 white beads and 18 black beads is*

$$
\begin{aligned}
N(12,18) &= G(12,18)/30 + G(6,9)/15 + G(4,6)/10 + G(2,3)/5 \\
&= 2\,882\,934 + 333 + 20 + 2 \\
&= 2\,883\,289\,.
\end{aligned}
$$

Of these, $2\,882\,934$ *are not periodic.*
333 *have the form XX where X has* $12/2 = 6$ *black beads and* $18/2 = 9$ *white beads.*
20 *have the form XXX where X has* $12/3 = 4$ *black beads and* $18/3 = 6$ *white beads.*
2 *have the form $XXXXXX$ where X has* $12/6 = 2$ *black beads and* $18/6 = 3$ *white beads.*
In each of the cases above, X is non-periodic.

To go from necklaces to partition cycles, note that $n = T_m + r$. If $r = 12$ and $m + 1 - r = 18$, then $m = 29$ and $n = T_{29} + 12 = 447$. It follows that $n = 447$ has $2\,883\,289$ cycles, of which $2\,882\,934$ are of length 30; 333 are of length 15; 20 are of length 10 and 2 are of length 5.

Examples of these can be reconstructed. The non-periodic cycles correspond to non-periodic sequences of 12 0's and 18 1's. Let one of these be X. Then construct P as $X+M$. For example,

$$\textit{If } X = (0,1,1,0,1,0,0,1,1,1,0,1,1,0,1,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0),$$
$$\textit{then } P = X + (0,1,2,\ldots,29) = (0,2,3,3,5,\ldots,27,28,29)$$

The same can be done for necklaces of the form XX, XXX, and $XXXXXX$. For example, in a necklace of the form $XXXXXX$, X could be a non-periodic sequence of 2 1's and 3 0's, such as

$$\begin{aligned} X &= (1,0,1,0,0) \\ P &= XXXXXX + (0,1,2,\ldots,29) \\ &= ([1,1,3,3,4],[6,6,8,8,9],\ldots,[26,26,28,28,29]). \end{aligned}$$

There is a symmetry between r and $m+1-r$. If $m+1-r=6$ and $r=4$, then m is still 9. The number of cycles for $n=T_9+4=49$ is the same as for $n=T_9+6=51$, since they both correspond to necklaces with 4 beads of one colour and 6 of another.

Theorem 58. *$n_1 = T_m + r$ and $n_2 = T_m + (m+1-r)$ have the same number of cycles.*

Proof. The numbers r and $m+1-r$ are interchangeable in the necklaces. Note that if $r=0$, then T_m and $T_m+m+1=T_{m+1}$ have the same number of standard cycles, namely 1. □

6 Goat restoration

The original goat problem is returned to with a different question. Will the goats ever end up in their original groups? Clearly, this is only possible if the original groups form a partition which lies in a cycle, for otherwise the same partition will not recur. For example, if there are 14 goats initially divided into two groups of 7, then those groups cannot recur.

Claim 59. *If there are n goats and their initial groups form a standard partition, then it is possible to restore them to their initial groups. If $n=T_m$, then this can be done in m steps; otherwise in $m+1$ steps.*

A proof will only be outlined, and a corresponding method. Note that rule R that was originally applied to the goats is not deterministic when considering individual goats, in the sense that it does not define which goat is selected from each group when reforming - any goat may be taken from its group. To restore goats to their original groups, each goat is coloured according to its original group, and the coloured groups must then be restored.

Definition 60. A group with only one colour is *monochrome*; otherwise, it is *polychrome*. A colour is *distributed* if no group has more than one instance of it. A colour is *restored* if all of its instances are in one group. The successive partitions are notated as G_j where G_0 is the original partition and G_k is the partition after k applications of rule R.

The modified rule R is as follows. R operates in the usual way on monochrome groups, taking one colour from each of these to the next new group. For polychrome groups, it takes a distributed unrestored colour from each group and places it in a new group, where it becomes restored. If there are two distributed colours, then it can take either one. Two modifications are required. If some group in G_0 has two more elements than the next smallest group, then R moves the least frequent colour from the polychrome group in G_1 to G_2. The reason for this is that a restored group would not otherwise have enough "padding" to make it to the end. The second modification is that if G_1 has two restored letters, then one of them must be moved.

A few examples are illustrated in Figure 15 - colours are indicated with letters, and new groups are indicated in **bold**.

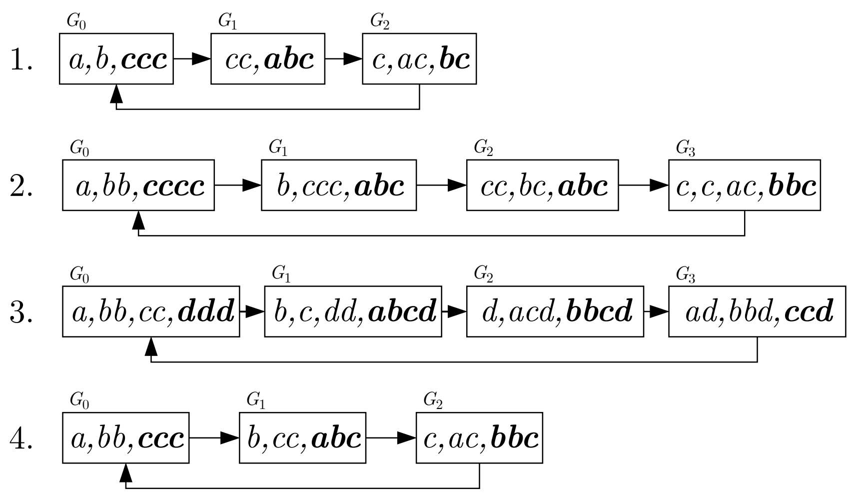

Figure 15: Goat restoration examples.

In the first example, both a and b are distributed in G_1 and restored, so one of them is moved to a new group. In the second example, the distributed letter b from G_1 was not moved to the new group in G_2, but a was moved instead. If b had been moved, then G_2 would have been (cc, ac, bbc) and both a and b would not have enough padding. In the third example there was a choice of applying R to b or to c in G_1. In the fourth example R applies three times without complications.

What remains to be proved is that it is always possible to apply R; that is, there is always a distributed letter after G_0. For $n = T_m$, restoring the goats requires m moves, since R applies once to each letter starting with G_0 - see the fourth example in the list above, where $n = T_3$. For $n = T_m + r < T_{m+1}$, the partition cycle requires $m + 1$ moves to reach the starting partition, so G_0 cannot be restored in fewer than $m + 1$ moves.

7 Summary

The goat problem introduced an operation R on groups of goats, in which each group sends one element to form a new group. The problem was abstracted to an operation H on a partition P of n, and the minimally repeating sequence $H^*(P)$ was introduced. A standard partition of $n = T_m + r < T_{m+1}$, where T_m and T_{m+1} are triangle numbers, was defined, and properties of cycles were explored. It turned out that if $r = 0$, then there is one standard cycle, of length 1. If $r > 1$, then the standard cycles are of length $m+1$ or a divisor of $m+1$. The operation $P - M$ gives a sequence X of r 1's and $m+1-r$ 0's, and the cycles of P correspond to rotations of X.

This let us equate cycle with necklaces formed from $P - M$ for standard P's. Some necklaces are periodic, corresponding to shorter cycles. Equations for the number of cycles for n were developed. Inverses of H were introduced and a method was given to determine the number of inverses of a given partition P, and how to determine them.

An application of this work showed that every cycle of standard partitions contains an entry point outside the cycle. Finally, the topic of goats was returned to, asking whether it is possible to return them to their original groups by repeated applications of R. We claimed that it is possible if the goats are initially divided into a standard partition, and not otherwise, and that it requires exactly $m+1$ divisions for $n = T_m + r$ if $r > 0$, and exactly m divisions for $n = T_m$. It has been left as an open problem whether only standard partitions form cycles, though this seems very plausible.

An interesting, almost amazing aspect of the topic is the role of triangle numbers in standard partitions and in cycles. Equally fascinating is the isomorphism of standard cycles with necklaces composed of r white beads and $m+1-r$ black beads.

Acknowledgements

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