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Solutions 1701–1710

Q1701 A school class consists entirely of twins: $2n^2 + 2n$ pairs of them, where $n \geq 2$. Together with the teacher, this means that there are $4n^2 + 4n + 1$ people in the class, so they can stand in a $2n + 1$ by $2n + 1$ square array. Prove that however they arrange themselves in this array, it will be possible to find $2n + 1$ of the children (excluding the teacher) in such a way that no two of the chosen children are standing in the same row, no two are standing in the same column, and no two are twins.

SOLUTION It is clear that there are many ways to choose $2n + 1$ children so that no two are in the same row or column; we have to prove that it is possible to do this in such a way that no two are twins. To make a selection in which no two children occupy the same row or column, we may proceed as follows.

- Choose a child from the same row as the teacher: this can be done in $2n$ ways.
- Specify the column of the child to be chosen from each remaining row. As there are $2n$ rows and $2n$ available columns, and the columns chosen must all be different, the number of ways to do this is $(2n)!$.

The total number of ways of making this selection is $(2n)(2n)!$.

Now count those of the above selections that also **do** contain a pair of twins. To do this,

- choose a pair of twins: this can be done in *at most* $2n^2 + 2n$ ways (possibly less, because if two twins are standing in the same row or column, they are not an allowable choice);
- choose a child from each remaining row, without repeating columns: there are *at most* $(2n - 1)!$ ways (possibly less, because some of these $(2n - 1)!$ choices may include the teacher).

So the number of choices which contain a pair of twins is, at the most,

$$(2n^2 + 2n)(2n - 1)! = (n + 1)(2n)! \,.$$

However, as $n \geq 2$, we have $n + 1 < 2n$, and so among all selections of children with no two in the same row or column, the number including a pair of twins is less than the total number; therefore, there must be a selection which does not include a pair of twins.

Comment. If $n = 1$ the result need not be true: for example, if four pairs of twins and a teacher are arranged as shown, then there is no selection of the required type.

$$\begin{matrix} 1 & 2 & 3 \\ 4 & \mathrm{T} & 1 \\ 2 & 3 & 4 \end{matrix}$$

Further comment. Note that in counting the number of selections including twins, we may have counted the same arrangement more than once. For example, suppose that the class includes twins Alexander and Ben, and another pair of twins Anna and Beatriz. Then any selection containing both pairs will have been counted once where Alexander and Ben are chosen first and Anna and Beatriz are "accidentally" chosen among the remaining $2n-1$ children; and once again in the opposite case. So the selection will have been counted twice, maybe even more. But this means that the true number of selections including twins is even less than we found above, so it does not affect our argument.

Q1702 Consider the sequence of numbers obtained by stringing together the digits of the positive integers, namely

$$1\,,\ 12\,,\ 123\,,\ 1234\,,\ 12345\,,\ 123456\,,\ldots$$
$$\ldots,\ 12345678910\,,\ 1234567891011\,,\ 123456789101112$$

and so on. Are any of these numbers multiples of 11? If so, find the smallest example.

SOLUTION Let s_n be the nth number in the sequence. We shall use congruence arithmetic. It is routine to check that $s_1, s_2, \ldots, s_9$ are not multiples of 11, and that $s_9 \equiv 5 \pmod{11}$. Now for $n = 10, 11, \ldots, 99$, we have

$$s_n = 100s_{n-1} + n \equiv s_{n-1} + n \pmod{11}$$

and so

$$s_n \equiv n + (n-1) + (n-2) + \cdots + 10 + s_9 \pmod{11}\,.$$

Multiplying both sides of the congruence by 2, adding up an arithmetic series and using the value of s_9 from above gives

$$2s_n \equiv (n+10)(n-9) + 10 \pmod{11}\,.$$

We want s_n to be a multiple of 11, that is,

$$(n+10)(n-9) + 10 \equiv 0 \pmod{11}\,.$$

Expanding and subtracting $11n$ (which counts as zero modulo 11) gives

$$n^2 - 10n - 80 \equiv 0 \pmod{11}\,;$$

completing the square and simplifying,

$$(n-5)^2 \equiv 105 \equiv 6 \pmod{11}\,.$$

But this is impossible, since 6 is not a square modulo 11 (see below for a reason why). So we calculate from above

$$s_{99} \equiv 99 + 98 + \cdots + 10 + s_9 \equiv 4 \pmod{11}$$

and proceed to consider appending three–digit numbers $n = 100, 101, \ldots, 999$. In this case, we have

$$s_n = 1000s_{n-1} + n \equiv -s_{n-1} + n \pmod{11}$$

and hence

$$s_{n+1} \equiv -s_n + n + 1 \equiv s_{n-1} + 1 \pmod{11}.$$

That is, each term s_n is one more (modulo 11) than the second previous term. It is now simple to find the values

$$s_{99},\ s_{100},\ s_{101}, \ldots \equiv 4,\ 8,\ 5,\ 9,\ 6,\ 10,\ 7,\ 0, \ldots \qquad \pmod{11},$$

and so the smallest number in the sequence which is a multiple of 11 is

$$s_{106} = 123456789101112 \cdots 99100 \cdots 106.$$

To confirm the statement that 6 is not a square modulo 11, note that every integer is congruent modulo 11 to one of $0, \pm1, \pm2, \pm3, \pm4, \pm5$, and the squares of these are

$$0^2, 1^2, 2^2, 3^2, 4^2, 5^2 \equiv 0, 1, 4, 9, 16, 25 \equiv 0, 1, 4, 9, 5, 3 \pmod{11},$$

and this list does not include 6.

Q1703 Solve the simultaneous equations

$$x\sqrt{x} + y\sqrt{y} = 183, \quad x\sqrt{y} + y\sqrt{x} = 182.$$

SOLUTION Expanding a cube gives

$$\left(\sqrt{x} + \sqrt{y}\right)^3 = \left(x\sqrt{x} + y\sqrt{y}\right) + 3\left(x\sqrt{y} + y\sqrt{x}\right) = 183 + 3 \times 182 = 729,$$

and hence

$$\sqrt{x} + \sqrt{y} = 9. \qquad (*)$$

If we add the two given equations and factorise the left–hand side, then we obtain

$$(x + y)\left(\sqrt{x} + \sqrt{y}\right) = 365,$$

and using the previous equation yields $x + y = \frac{365}{9}$. Squaring $(*)$ therefore gives

$$x + 2\sqrt{xy} + y = 81$$

so

$$\sqrt{xy} = \frac{81 - \frac{365}{9}}{2} = \frac{182}{9}.$$

Now consider the quadratic polynomial $\left(t - \sqrt{x}\right)\left(t - \sqrt{y}\right)$. Its roots are $\sqrt{x}$ and $\sqrt{y}$; but expanding gives

$$\left(t - \sqrt{x}\right)\left(t - \sqrt{y}\right) = t^2 - \left(\sqrt{x} + \sqrt{y}\right)t + \sqrt{xy} = t^2 - 9t + \tfrac{182}{9},$$

and solving in the usual way finds the roots $t = \frac{13}{3}, \frac{14}{3}$.
Therefore, x and y are $\frac{169}{9}$ and $\frac{196}{9}$, in either order.

Solutions received. Jason Zimba, New York, sent four different ways of solving this problems! Solutions were also submitted by Soham Dutta, by Shivam Mokashi and Abhinava Vidyalaya, India; and by Kyumin Nam, Incheon Shinjeong Middle School, South Korea.

Q1704 The Chinese red packet is a tradition that goes back over 2000 years as a way of gifting money to children for the Lunar New Year, and also at weddings. The problem below requires a blend of luck and skill.

A player is faced with 1001 red packets of which 5 lucky packets contain money (and the other 996 are empty). In each round, the player selects any number of the envelopes and divides them into a maximum of 5 piles. The player is then told the number of lucky packets in each pile. The objective is to obtain 5 piles of red packets where each pile has exactly one lucky packet. What is the maximum number of rounds required?

SOLUTION Let n_i be the number of packets selected in round i. Split the n_i packets into five piles, as nearly as possible equal in number. Any pile which is identified as containing one lucky packet can be set aside for later; any pile with no lucky packet can be ignored; and the player will need to continue the game with any piles having two or more lucky packets. Since the chosen packets are split into 5 piles and there are 5 lucky packets, at most two of the piles will satisfy this requirement. How many envelopes are in each pile? If the number of packets at the beginning of a round is a multiple of 5, say $5k$, then every pile will contain k packets. If it is $5k+1$, then one pile will contain $k+1$ packets and each other pile k. If it is $5k+2$, $5k+3$ or $5k+4$ then at least two piles will contain $k+1$ packets. Therefore, in the unluckiest outcome, the total number of packets n_{i+1} in the next round is given by

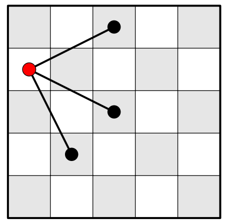
$$n_{i+1} = \begin{cases} 2k & \text{if } n_i = 5k \\ 2k+1 & \text{if } n_i = 5k+1 \\ 2k+2 & \text{otherwise.} \end{cases}$$

Starting with $n_1 = 1001$ and applying this formula gives the maximum numbers of packets to be considered in subsequent rounds:

$$1001 \to 401 \to 161 \to 65 \to 26 \to 11 \to 5\,.$$

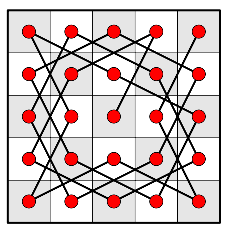
With only 5 packets remaining, the procedure is now certain to finish successfully. So the maximum number of rounds is 7.

Q1705 In the game of chess, a knight can move from its current square to any square reached by going two squares horizontally or vertically, then one square in a perpendicular direction. The diagram



shows all the moves available on a 5×5 chessboard to a knight on the white square marked with a red dot. Since a single move takes a knight from a white square to a black square, or *vice versa*, no two of the 13 black squares are separated by a single knight's move. Is there any other choice of 13 squares on a 5×5 board for which this is true?

SOLUTION There is no other choice of 13 squares on a 5×5 chessboard such that no two are separated by a knight's move. To prove this, observe that the diagram



shows a path formed of knight's moves which includes all 25 squares on the board.

To obtain a set of squares in which no two are separated by a knight's move, we cannot take two consecutive squares along this path. So the only way to obtain 13 squares is to take the first, third, fifth and so on of the 25 squares. Looking carefully at the diagram shows that these are precisely the black squares on the board, and there is no other way to satisfy the requirement.

Q1706 Arrange a hundred digits (digits are $0, 1, \ldots, 9$) on a circle in such a way that, reading clockwise, every one of the pairs $00, 01, \ldots, 99$ occurs once each.

SOLUTION We show how to do this construction inductively for n^2 digits, where the allowable digits are $0, 1, \ldots, n-1$. Clearly,

$$1100$$

does the job for $n = 2$. (It's not a circle but that would take a lot of printing space later on! You can imagine that the end is joined to the beginning so as to form a circle.) If we have already a solution for some specific n, then it must contain the pair 00 somewhere. In between these two zeros insert

$$n\,,\ n\,,\ n-1\,,\ n\,,\ n-2\,,\ n\,,\ n-3\,,\ \ldots\,,\ n\,,\ 0\,. \qquad (*)$$

Then it is easy to see that every pair xy where $x = n$ or $y = n$ or both occurs in this insertion, except for $0n$. Also, every pair xy where neither x nor y is n occurs in the previous iteration, except that 00 has been split by the insertion. However, since the insertion occurs between two 0s, both these exceptions do occur after all: $0n$ where the insertion begins, and 00 where it finishes. Therefore, the combined arrangement includes all pairs of digits from 0 to n. Specifically, for $n = 10$, one possible solution is

$$1102212033231304434241405545352515066564636261607 7\ldots$$
$$\ldots 6757473727170887868584838281809989796959493929190 0\,.$$

Comment. Many solutions are possible: for example, in $(*)$, the digits $n-1, n-2, \ldots, 1$ could be permuted among themselves in any way.

Q1707 Consider all numbers that can be formed by choosing eleven different positive integers whose sum is 82 and finding the product of the eleven integers. For example, one of the products to be considered is

$$1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 27\,.$$

Find the greatest common divisor (highest common factor) of all these products.

SOLUTION The required greatest common divisor is 24. To prove this, we begin by noting that every one of the products is a multiple of 3, for if any product of 11 different positive integers is not a multiple of 3, then the sum of the numbers is at least

$$1 + 2 + 4 + 5 + 7 + 8 + 10 + 11 + 13 + 14 + 16 = 91$$

and cannot be 82. Similarly, every product must be a multiple of 8, for if not, then the numbers include at most two even numbers, and their sum is at least

$$1 + 2 + 3 + 4 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 87\,.$$

Therefore, the greatest common divisor of all products is at least 24. However, we can find cases in which the product does not have 16 as a factor, for example,

$$1 + 2 + 3 + 5 + 6 + 7 + 9 + 10 + 11 + 13 + 15 = 82\,;$$

so 16 is not a factor of *all* the products under consideration, and is therefore not a factor of the greatest common divisor. Likewise, we can find cases where the product does not have 9 as a factor, for example,

$$1 + 2 + 3 + 4 + 5 + 7 + 8 + 10 + 11 + 14 + 17 = 82\,,$$

and where the product does not have 5 as a factor, for example,

$$1 + 2 + 3 + 4 + 6 + 7 + 8 + 9 + 11 + 12 + 19 = 82\,,$$

and, finally, where the product does not have any prime factor greater than 5, for example,

$$1 + 2 + 3 + 4 + 5 + 6 + 8 + 9 + 10 + 16 + 18 = 82\,.$$

Therefore, the greatest common divisor cannot be greater than 24, so it is exactly 24.

Q1708 How many paths of length $m+n+2$ are there from $(0,0)$ to (m,n) on an $m\times n$ grid, if the path may never visit the same grid point more than once?

SOLUTION Such a path must consist of m moves to the right and n moves up, together with **either** an extra move right and one left, **or** an extra move up and one down. In the first case, the left move cannot have a right move immediately before or after it as this would mean some point was visited twice; the left cannot occur before the first right as it would take the path off the grid; and it cannot occur after the last right as it would have come from off the grid. A similar argument applies to the case where there is an extra up and a down move; therefore, a path such as we are seeking must consist of one of the following mutually exclusive possibilities:

- $m+1$ single moves right, $n-2$ single moves up, 1 sequence of three moves up–left–up;
- $m-2$ single moves right, $n+1$ single moves up, 1 sequence of three moves right–down–right.

To count the first type, we begin by arranging $m+2$ letters R and $n-2$ letters U: this can be done in $C(m+n,n-2)$ ways. Then we replace one of the letters R by ULU: this can be done for any of the $m+2$ letters R except the first or last, so there are m options. A similar argument for the second case gives the total number of paths as

$$m\binom{m+n}{n-2}+n\binom{m+n}{m-2}.$$

Q1709 Find the largest 8–digit number which uses the digits $1,2,3,4,5,6,7,8$ once each and is a multiple of 101.

SOLUTION Let N be an 8–digit number formed from the digits $1,2,3,4,5,6,7,8$, and write N as a concatenation of two–digit numbers $n_1n_2n_3n_4$. We have

$$N=10^6n_1+10^4n_2+10^2n_3+n_4=(101-1)^3n_1+(101-1)^2n_2+(101-1)n_3+n_4\,,$$

and if we expand this, then we find that N is a multiple of 101, plus a remainder $R=-n_1+n_2-n_3+n_4$. So N is a multiple of 101 if and only if R is a multiple of 101.

Since we want the largest value of N, we try taking $n_1=87$ and $n_2=65$, that is, $N=8765dddd$. If we can find solutions in this case, then they cannot be exceeded by any other solutions. This gives $R=n_4-n_3-22$, which must be a multiple of 101. Since n_3 and n_4 consist of the digits $1,2,3,4$, we have

$$12-43-22\le R\le 43-12-22\,,$$

that is, $-53\le R\le 9$. As R is a multiple of 101, it must be zero, and we have $n_4=n_3+22$. Since the digits of n_3,n_4 are $1,2,3,4$ and we want the largest possible n_3, we have $n_3=21$ and $n_4=43$. So the required number is $N=87652143$.

Comment. If you are familiar with *modular arithmetic*, then you will be able to simplify some of the above working.

Q1710 Prove that if a, b, c are positive and $a + b + c = 3$, then

$$\frac{a^2 + b^2}{\sqrt{ab}} + \frac{b^2 + c^2}{\sqrt{bc}} + \frac{c^2 + a^2}{\sqrt{ca}} \geq 6\,.$$

SOLUTION From the Arithmetic Mean–Geometric Mean Inequality, we have

$$\sqrt{ab} \leq \frac{a + b}{2}\,.$$

Using this together with similar results involving b, c and c, a, we have

$$\frac{a^2 + b^2}{\sqrt{ab}} + \frac{b^2 + c^2}{\sqrt{bc}} + \frac{c^2 + a^2}{\sqrt{ca}} \geq 2\frac{a^2 + b^2}{a + b} + 2\frac{b^2 + c^2}{b + c} + 2\frac{c^2 + a^2}{c + a}\,. \qquad (*)$$

Also $a^2 + b^2 \geq 2ab$, so $2(a^2 + b^2) \geq a^2 + b^2 + 2ab = (a + b)^2$. Therefore,

$$2\frac{a^2 + b^2}{a + b} \geq a + b\,.$$

Substituting this and similar results into $(*)$ yields

$$\frac{a^2 + b^2}{\sqrt{ab}} + \frac{b^2 + c^2}{\sqrt{bc}} + \frac{c^2 + a^2}{\sqrt{ca}} \geq (a + b) + (b + c) + (c + a) = 2(a + b + c) = 6\,.$$

Solutions to this problem were received from Kunihiko Chikaya, Tokyo, Japan; from Soham Dutta; from Henry Ricardo, New York, USA; and from Kyumin Nam, Incheon Shinjeong Middle School, South Korea.