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Curious properties of iterative sequences

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1 Introduction

In this study, several interesting iterative sequences were investigated.

First, we define the iterative sequences. Let $\mathbb{N}$ be the set of natural numbers and fix a function f from $\mathbb{N}$ to $\mathbb{N}$. An iterative sequence starts with $n \in \mathbb{N}$ and continues as

$$f(n),\ f^2(n) = f(f(n)),\ \ldots,\ f^{m+1}(n) = f(f^m(n)),\ \ldots .$$

We then search for interesting features in this sequence.

The study of iterative sequences is a useful topic for professional or amateur mathematicians if they choose the proper problem.

We begin with the Collatz Conjecture, which is one of the most well-known unsolved problems concerning iterative sequences. A mathematician Lothar Collatz presented the following prediction 86 years ago:

Prediction 1. *For any natural number n, if we apply the function in (1) repeatedly to n, then we eventually reach the number 1.*

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even;} \\ 3n+1 & \text{if } n \text{ is odd.} \end{cases} \tag{1}$$

Collatz Conjecture was independently proposed by other mathematicians Helmut Hasse, Shizuo Kakutani, Bryan Thwaites, and Stanisław Ulam.

Example 1. The function f is here applied to several numbers.
(i) If we start with the number $n = 67$ and iteratively apply the function f of (1), then we obtain the following sequence:

$$67, 202, 101, 304, 152, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1 .$$

(ii) Now starting with $n = 36$, we obtain

$$36, 18, 9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1 .$$

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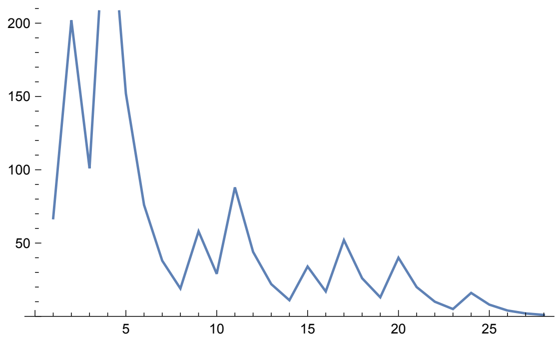

Figure 1: Graph of sequence in (i).

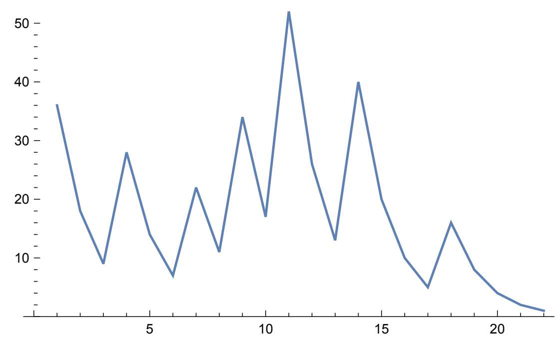

Figure 2: Graph of sequence in (ii).

The calculations of (i) and (ii) were carried out using the program [15].

Although many mathematicians have studied this problem for 86 years, there is still no proof nor counterexample for this prediction. Because the Collatz Conjecture is simple and well-known, it has attracted the attention of many people, and many amateur mathematicians spend a considerable amount of time on it.

It is good to appreciate this conjecture; however, it is not wise for most people to attempt to resolve it. The eminent mathematician Paul Erdős said about the Collatz Conjecture: "Mathematics may not be ready for such problems." For these and other details on the Collatz Conjecture, see [1].

In the remainder of this article, we study iterative sequences, such as Kaprekar's Routine, the digit factorial process, and the digit power process. We also present new variants of Kaprekar's Routine.

Regarding the digit power and digit factorial processes, two of the present authors have already presented some results in [2, 3]. In this article, we study these processes from a new perspective. For further information on iterative sequences, see [4].

2 Kaprekar's Routine

Kaprekar [5] discovered an interesting property of an iterative sequence, which was subsequently named *Kaprekar's Routine*. In this iterative sequence, we start with a four-digit number whose four digits are distinct. If we subtract the highest and lowest numbers constructed from these four digits and continue in this fashion, then we eventually obtain 6174, called *Kaprekar's Constant*.

Example 2. Kaprekar's Routine is illustrated below.
(i) If we start with $n = 1234$, then we obtain the sequence

$$4321 - 1234 = 3987, \qquad 8730 - 0378 = 8352, \quad \text{and} \quad 8532 - 2358 = 6174\,.$$

Since $7641 - 1467 = 6174$, we obtain the same number, 6174.
(ii) If we start with $n = 1001$, then we obtain the sequence

$$1100 - 11 = 1089, \quad 9810 - 189 = 9621, \quad 9621 - 1269 = 8352, \quad \text{and} \quad 8532 - 2358 = 6174\,.$$

Again, we obtain the number 6174.

For more details of Kaprekar's Routine, see [6]. There seem to be many things that can be discovered about Kaprekar's Routine – see Section 3.

Example 3. The authors present a Python program for Kaprekar's Routine. This program searches for four-digit numbers that do not converge to Kaprekar's Constant.

The program is available at [15, blob/main/Kaprekar]. The output is as follows:

```
Four-digit numbers that do not converge to Kaprekar's constant (6174):
[1111, 2222, 3333, 4444, 5555, 6666, 7777, 8888, 9999]
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3 Variants of Kaprekar's Routine

Here, we studied a few variants of Kaprekar's Routine.

3.1 The case of base 10

Definition 4. Let n, u, v be natural numbers. Write n using digits and list all natural numbers obtained by permuting these digits. Denote by α_u and β_v the u-th largest number and v-th smallest number in this list, respectively. Define $K_{u,v}(n)$ to be the difference $\alpha_u - \beta_v$, and let $K_{u,v}^t(n)$ to be the t-th iteration of this function applied to n.

Example 5. Consider the triple

$$(n, u, v) = (594, 2, 1).$$

By permuting the digits of n, we obtain the following list of natural numbers written in descending order:

$$(954, 945, 594, 549, 495, 459)\,.$$

We obtain

$$\alpha_u = \alpha_2 = 945 \quad \text{and} \quad \beta_v = \beta_1 = 459\,.$$

Therefore,

$$K_{2,1}^1(594) = K_{2,1}(594) = \alpha_2 - \beta_1 = 945 - 459 = 486\,.$$

Example 6. Here are interesting results regarding the convergences of the sequences $\{K_{u,v}^t(n) : t = 1, 2, \dots\}$ where (n, u, v) is fixed. These were obtained by using the program [15, blob/main/Kaprekar/base10].

(a) For any 3-digit number n, $\{K_{2,2}^t(n) : t = 1, 2, \dots\}$ reaches the fixed point 450.

(b) For any 4-digit number n, $\{K_{3,1}^t(n) : t = 1, 2, \dots\}$ reaches the fixed point 4995.

(c) For any 5-digit number n, $\{K_{3,2}^t(n) : t = 1, 2, \dots\}$ reaches the fixed point 49995.

(d) For any 5-digit number n, $\{K_{4,1}^t(n) : t = 1, 2, \dots\}$ reaches the fixed point 62748.

(e) For any 4-digit number n, $\{K_{1,2}^t(n) : t = 1, 2, \dots\}$ reaches one of the fixed points $9045, 4995, 4997$.

3.2 The case of base 2

In this subsection, we treat all of the numbers in base 2.

Definition 7. We choose any natural number n in base 2. We sort the digits of n in descending order, and let α_2 be the second-largest number. Subsequently, we sort the digits of n in ascending order and let β_2 be the second-smallest number. These numbers may have leading zeros, which are retained. Then, we subtract $\alpha_2 - \beta_2$ to generate the next number, $K_{2,2}(n)$. The t-th iteration of this function is denoted by $K_{u,v}^t(n)$.

Conjecture 8. *Let n be a natural number, such that $m \geq 3$.*
(i) When the binary digit length of n is $2m$, the sequence $\{K_{2,2}^t(n) : t = 1, 2, \dots\}$ reaches the fixed point $2^{2m} - 3 \times 2^m + 1$.
(ii) When the binary digit length of n is $2m+1$, the sequence $\{K_{2,2}^t(n) : t = 1, 2, \dots\}$ enters a loop of the two numbers $2^{2m+1} - 7 \times 2^m + 2^{m-2} + 1$ and $2^{2m+1} - 7 \times 2^m + 10 \times 2^{m-2} + 1$.

For the numbers with digit lengths of $6, 7, 8, 9$ in base 2, Conjecture 8 is valid, as presented in Table 1. The computer program [15, blob/main/Kaprekar/base2] was used to perform these calculations.

Table 1: Fix points and loops for the binary iteration function $K_{u,v}^t(n)$

Base	Digit length	fixed points	loops
2	4	101, 10	none
	5	none	{110,1111}
	6	$101001 = 2^{2\cdot 3} - 3 \cdot 2^3 + 1$	none
	7	none	$\{1001011, 1011101\}$ $= \{2^{2\cdot 3+1} - 7 \cdot 2^3 + 10 \cdot 2^{3-2} + 1,$ $2^{2\cdot 3+1} - 7 \cdot 2^3 + 2^{3-2} + 1\}$
	8	$11010001 = 2^{2\cdot 4} - 3 \cdot 2^4 + 1$	none
	9	none	{110010101,110111001} $\{2^{2\cdot 4+1} - 7 \cdot 2^4 + 10 \cdot 2^{4-2} + 1,$ $2^{2\cdot 4+1} - 7 \cdot 2^4 + 2^{4-2} + 1\}$

4 Digits factorial process

This process was studied in [3, 4]. We present new results in Lemma 15, Theorem 16, and Lemma 23.

Define the following function of each natural number n:

$$\mathbf{dfp}(n) = \sum_{k=1}^{m} n_k!\,,$$

where $n_1, n_2, \dots, n_m$ are the decimal digits of n. Starting with any natural number n, we repeatedly apply $\mathbf{dfp}$ to generate the sequence of integers $\mathbf{dfp}(n), \mathbf{dfp}^2(n), \dots$.

Example 9. The first few terms of the sequence $\{\mathbf{dfp}^t(123) : t = 1, 2, \dots\}$ are

$$\begin{aligned}\mathbf{dfp}(123) &= 1! + 2! + 3! = 9\,,\\ \mathbf{dfp}^2(123) &= \mathbf{dfp}(9) = 9! = 362880\,,\\ \mathbf{dfp}^3(123) &= \mathbf{dfp}(362880) = 3! + 6! + 2! + 8! + 8! + 0! = 81369\,.\end{aligned}$$

Lemma 10. *Below are some fixed points and loops of* $\mathbf{dfp}$*:*

(i) $\{1\}$, $\{2\}$, $\{145\}$, $\{40585\}$;

(ii) $\{871, 45361\}$, $\{872, 45362\}$;

(iii) $\{169, 363601, 1454\}$.

Proof. Verify computationally by applying $\mathbf{dfp}$ to each of the numbers above. □

Remark 11. The loops in Lemma 10 are registered as [7, 8, 9] in the On-Line Encyclopedia of Integer Sequences.

In [4], Lehmann wrote that the above-mentioned loops are the only loops in natural numbers smaller than 2000000. However, we can prove that these are the only loops for all natural numbers.

Lemma 12. *For each natural number* $m \geq 2$*,*

$$\frac{10^m - 1}{m \times 10!} < \frac{10^{m+1} - 1}{(m+1) \times 10!}.$$

Proof. Since $m \geq 2$,

$$\frac{m+1}{m} < 2 < 10 < \frac{10^{m+1} - 1}{10^m - 1}\,.$$

The desired inequality follows. □

Lemma 13. *For each natural number* $m \geq 8$*,*

$$\frac{10^m - 1}{10} > m \times 9!\,. \tag{2}$$

Proof. By Lemma 12, the sequence

$$\left\{\frac{10^m - 1}{m \times 10!} : m = 2, 3, 4, \dots\right\}$$

is increasing. Also, $\dfrac{10^m - 1}{m \times 10!} \approx 3.44 > 1$ for $m = 8$, so

$$\frac{10^m - 1}{m \times 10!} > 1$$

for all $m \geq 8$. □

Lemma 14. *For each natural number $n \geq 10^7$,*

$$\mathbf{dfp}(n) < n\,.$$

Proof. For $n \geq 10^7$, there exists a natural number $m \geq 7$ such that

$$10^m \leq n \leq 10^{m+1} - 1\,. \tag{3}$$

Therefore, Lemma 13 and the definition of $\mathbf{dfp}$ imply that

$$\mathbf{dfp}(n) \leq (m+1)9! \leq \frac{10^{m+1}-1}{10} < 10^m \leq n\,.$$

□

Lemma 15. *If we start with an arbitrary natural number n and repeatedly apply $\mathbf{dfp}$, then we obtain a sequence of numbers that eventually forms a loop or a fixed point.*

Proof. By Lemma 14, we have $\mathbf{dfp}(n) < n$ for $n \geq 10^7$. Also, if $n < 10^7$, then

$$\mathbf{dfp}(n) \leq 7 \times 9! = 2540160 < 10^7\,.$$

Therefore, for all natural numbers n, the sequence $\{\mathbf{dfp}^t(n) : t = 1, 2, \dots\}$ consists of numbers all smaller than 10^7. This is a finite set, so the numbers $\mathbf{dfp}^t(n)$ will at some point repeat themselves, either as a loop or a fixed point. □

By Lemma 15, the sequence $\{\mathbf{dfp}^t(n) : t = 1, 2, \dots\}$ ends in a loop or a fixed point for each natural number n. However, this lemma does not provide any information regarding the properties of these loops or fixed points. This information is given in the following theorem.

Theorem 16. *If we start with an arbitrary natural number n and repeatedly apply $\mathbf{dfp}$, then we obtain a sequence of numbers that eventually forms one of the loops or fixed points in Lemma 10.*

We prove this theorem in Section 6.

5 Digits power process

This process was first studied by some of the present authors in [2].

For each natural number n, we define

$$\mathbf{dpp}(n) = \sum_{k=1}^{m} n_k^{n_k}\,,$$

where $n_1, n_2, \dots, n_m$ are the decimal digits of n. We repeatedly apply the function $\mathbf{dpp}$ to generate the sequence of integers $\mathbf{dpp}(n), \mathbf{dpp}^2(n), \dots$.

Example 17. *The first few terms of the sequence* $\{\mathbf{dpp}^t(123) : t = 1, 2, \ldots\}$ *are*

$$\begin{aligned}
\mathbf{dpp}(123) &= 1^1 + 2^2 + 3^3 = 32\,, \\
\mathbf{dpp}^2(123) &= \mathbf{dpp}(32) = 3^3 + 2^2 = 31\,, \\
\mathbf{dpp}^3(123) &= \mathbf{dpp}(\mathbf{dpp}^2(123)) = \mathbf{dpp}(31) = 3^3 + 1^1 = 28\,.
\end{aligned}$$

Lemma 18. *For each natural number* $m \geq 2$*,*

$$\frac{10^m - 1}{m \times 9^9} < \frac{10^{m+1} - 1}{(m+1) \times 9^9}\,.$$

Proof. Since $m \geq 2$,

$$\frac{m+1}{m} < 2 < 10 < \frac{10^{m+1} - 1}{10^m - 1}\,.$$

The desired inequality follows. □

Lemma 19. *For each natural number* $m \geq 10$*,*

$$\frac{10^m - 1}{10} > m \times 9^9\,.$$

Proof. By Lemma 18, the sequence

$$\left\{\frac{10^m - 1}{m \times 9^9} : m = 2, 3, 4, \ldots\right\}$$

is increasing. Also, $\dfrac{10^m - 1}{m \times 9^9} \approx 2.58 > 1$ for $m = 10$, so

$$\frac{10^m - 1}{m \times 9^9} > 1$$

for natural numbers $m \geq 10$. □

Lemma 20. *For each natural number* $n \geq 10^{10}$*,*

$$\mathbf{dpp}(n) < n\,.$$

Proof. For $n \geq 10^{10}$, there exists a natural number $m \geq 10$ such that

$$10^m \leq n < 10^{m+1} - 1\,.$$

Therefore,

$$\mathbf{dpp}(n) \leq (m+1)9^9 < \frac{10^{m+1} - 1}{10} < 10^m \leq n\,.$$

□

Example 21. Here are a few fixed points and loops for $\mathbf{dpp}$:

$\{1\},\ \{3435\}$;
$\{421845123, 16780890\}$;
$\{16777500, 2520413, 3418\}$;
$\{809265896, 808491852, 437755524, 1657004, 873583, 34381154, 16780909, 792488396\}$;
$\{791621579, 776537851, 19300779, 776488094, 422669176, 388384265, 50381743, 17604196, 388337603, 34424740, 824599\}$;
$\{793312220, 388244100, 33554978, 405027808, 34381363, 16824237, 17647707, 3341086, 16824184, 33601606, 140025, 3388, 33554486, 16830688, 50424989, 791621836, 405114593, 387427281, 35201810, 16780376, 18517643, 17650825, 17653671, 1743552, 830081, 33554462, 53476, 873607, 18470986, 421845378, 34381644, 16824695, 404294403, 387421546, 17651084, 17650799, 776537847, 20121452, 3396, 387467199\}$;
$\{1583236420, 16827317, 18470991, 792441996, 1163132183, 16823961, 404291050, 387424134, 17601586, 17697199, 1163955211, 387473430, 18424896, 421022094, 387421016, 17647705, 2520668, 16873662, 17740759, 389894501, 808398820, 454529386, 404251154, 7025, 826673, 17694102, 388290951, 808398568, 454579162, 388297455, 421805001, 16780606, 17740730, 2470915, 388247419, 421799008, 792442000, 388244555, 33564350, 53244, 3668, 16870555, 17656792, 389164017, 405068190, 404247746, 1694771, 389114489, 808395951, 808401689, 437799052, 776491477, 390761830, 405067961, 388340728, 51155506, 59159, 774847229, 406668854, 33698038, 421021659, 387470537, 19251281, 404200841, 16777992, 777358268, 36074873, 18471269, 405068166, 16920568, 404294148, 404198735, 405024914, 387424389, 421799034, 775665066, 1839961, 791664879, 793358849, 809222388, 437752177, 3297585, 405027529, 388250548, 50338186, 33604269, 387514116, 17650826, 17697202, 389114241, 404198251, 404201349, 387421291, 405021541, 6770, 1693743, 388290999\}$.

Remark 22. Five of the above loops are registered by one of present authors in the On-Line Encyclopedia of Integer Sequences; see [10, 11, 12, 13, 14].

Lemma 23. *If we start with an arbitrary natural number n and apply $\mathbf{dpp}$ repeatedly, then we obtain a sequence of numbers that eventually forms a loop or a fixed point.*

Proof. By Lemma 20, we have $\mathbf{dpp}(n) < n$ for $n \geq 10^{10}$. Also, if $n < 10^{10}$, then

$$\mathbf{dpp}(n) \leq 10 \times 9^9 = 3874204890 < 10^{10} .$$

Therefore, for all natural numbers n, the sequence $\{\mathbf{dpp}^t(n) : t = 1, 2, \ldots\}$ consists of numbers all smaller than 10^{10}. This is a finite set, so the numbers $\mathbf{dpp}^t(n)$ will at some point repeat themselves, either as a loop or a fixed point. □

Theorem 24. *If we start with any natural number n and repeatedly apply the function* **dpp**, *then we obtain a sequence of numbers that eventually forms one of the loops or fixed points in Example 21.*

We prove this theorem in the next section.

6 Proofs by computers

We prove Theorems 16 and 24 by computational analysis. By Lemma 15, we need to prove Theorem 16 for any natural number $n < 10^7$. Also, by Lemma 23, we need to prove Theorem 24 for for any natural number $n < 10^{10}$. However, we present a method to simplify these calculations.

Lemma 25. *Let r be a natural number. The number of ways in which to choose an unordered collection of r not necessarily distinct numbers from the set $\{1, 2, \dots, 9\}$ is*

$$\binom{r+8}{r}. \tag{4}$$

Proof. The number of ways in which to choose an unordered collection of r not necessarily distinct elements from a set of n elements if repetitions are allowed is

$$\binom{n+r-1}{r}.$$

This can be shown by the dots and lines method of counting. □

Lemma 26.
(i) *To prove Theorem 16, we only need to consider* 11439 *numbers.*
(ii) *To prove Theorem 24, we must consider* 92377 *numbers.*

Proof. In the digit factorial process, the order of the digits is not relevant to the creation of the next number. Also, since $0! = 1! = 1$, we treat 0 and 1 as the same number.

(i) By Lemma 15, we only need to consider numbers smaller than 10^7, that is, numbers with fewer than 8 decimal digits. Therefore, Lemma 25 implies that the total number of numbers to consider is

$$\sum_{r=1}^{7} \binom{r+8}{r} = 11439\,.$$

(ii) By Lemma 23, we only need to consider numbers smaller than 10^{10}, that is, numbers with fewer than 11 decimal digits. Therefore, Lemma 25 implies that the total number of numbers to consider is

$$\sum_{r=1}^{10} \binom{r+8}{r} = 92377\,.$$

□

Computer Calculation 27. *We prove Theorems 16 and 24 via calculations using the program [15, tree/main/others] which is based on Lemma 26.*

7 Prospects for future research

Iterative functions are a good topic for high school or undergraduate research; however, studying well-known unsolved problems such as the Collatz Problem is not a good idea. A good way to conduct research is to change some parts of the original problem, such as that in Section 3. We can also use `XOR` instead of $+$ in digit factorial or digit power processes. After studying this process, please feel welcome to see the program [15, tree/main/others] under Digits%20Factorial%20XOR.

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