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# Malfatti's problem for an equilateral triangle

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## 1 Introduction

Gian Francesco Malfatti (1731–1807) was an Italian mathematician (for more information about him see [5]) who proposed the problem of carving out three cylindrical columns of a marble right-triangular prism in 1803 [4]. The three columns as well as the prism have the same altitude, and the material waste must be as small as possible. Thus, the columns must have maximal sum of their volumes. If we focus on the triangular base of the prism, then this problem can be transformed into the following plane geometry problem: find three non-overlapping circles packed inside a given triangle with maximal total area.

## 2 Two methods of solving

Malfatti assumed the best solution of this so-called *Malfatti's problem* or *Malfatti's marble problem* is three circles which are tangent to each other and also to two sides of the triangle. These circles are referred to as the *Malfatti circles* nowadays.

Later, mathematicians studied other ways of packing circles inside a triangle. One of them is a greedy algorithm consisting of a series of three steps. In the first step, we construct the circle of maximal radius (area) inside the triangle. It touches three sides of triangle (i.e., it is the incircle of the triangle). In the second step, we draw the circle of maximal radius which is inside the triangle and which does not overlap the first circle. Finally, in the third step, we find the circle of maximal radius which is inside the triangle and which overlaps neither the first nor second circle.

Let us look at both of these methods for representative triangles.

## 3 Isosceles triangles

In the case of the isosceles triangle which is shown in Figures 1 and 2 (the triangle with a small angle at the apex against the base), the Malfatti circles (see Figure 1) cover only approximately 35.7% of the area of the triangle. In the event of a greedy algorithm (see Figure 2), it is approximately 54.5%. Thus, the greedy arrangement is better and the difference is 18.8%.

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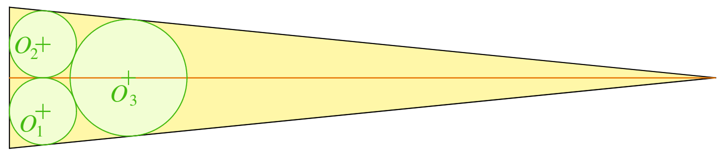


**Figure 1.**

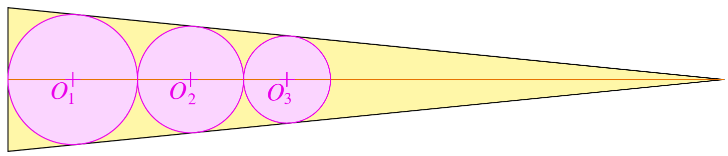


**Figure 2.**

If we consider the isosceles triangle in Figures 3 and 4 (the triangle has the same altitude as the triangle in the previous case and its base has twice the length), then the Malfatti circles (see Figure 3) cover approximately 55.2% of the area of the triangle. In the event of gradually packing circles of total maximum area inside the triangle (see Figure 4), they cover approximately 69.9% of the area of the triangle. So, this time the greedy algorithm is more optimal again but the difference, which is equal to 14.7%, is smaller than in the previous case.

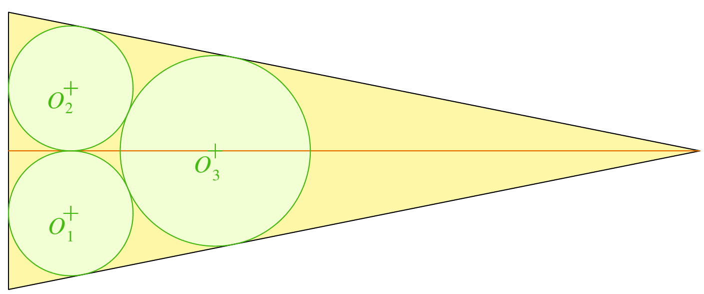


**Figure 3.**

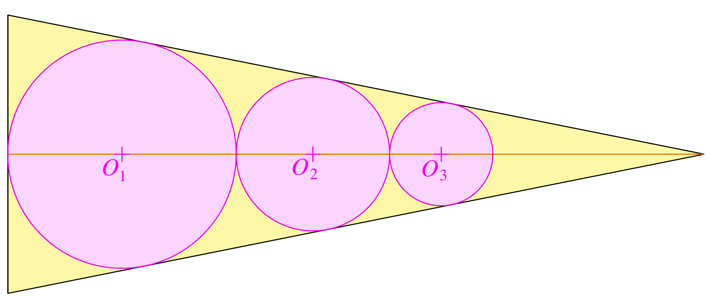


**Figure 4.**

The question naturally arises as to how it will turn out in the case of an equilateral triangle. Solving this problem with the help of analytical geometry, which can be done by high school students, is the main content of this article.

## 4 Equilateral triangles

Thus, consider an equilateral triangle $ABC$ and let $a = |BC| = |CA| = |AB|$ denote the length of its sides.

Set up a Cartesian coordinate system with $A$ at the origin and the positive direction of the $x$-axis which coincides with the ray $AB$ (see Figure 5). Then the vertices of the triangle $ABC$ have the following coordinates:

$$A = (0,0)\,, \quad B = (a,0) \quad \text{and} \quad C = \left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right).$$

The area $S_\triangle$ of the triangle under consideration is

$$S_\triangle = \frac{a \cdot \frac{\sqrt{3}a}{2}}{2} = \frac{\sqrt{3}a^2}{4}\,.$$

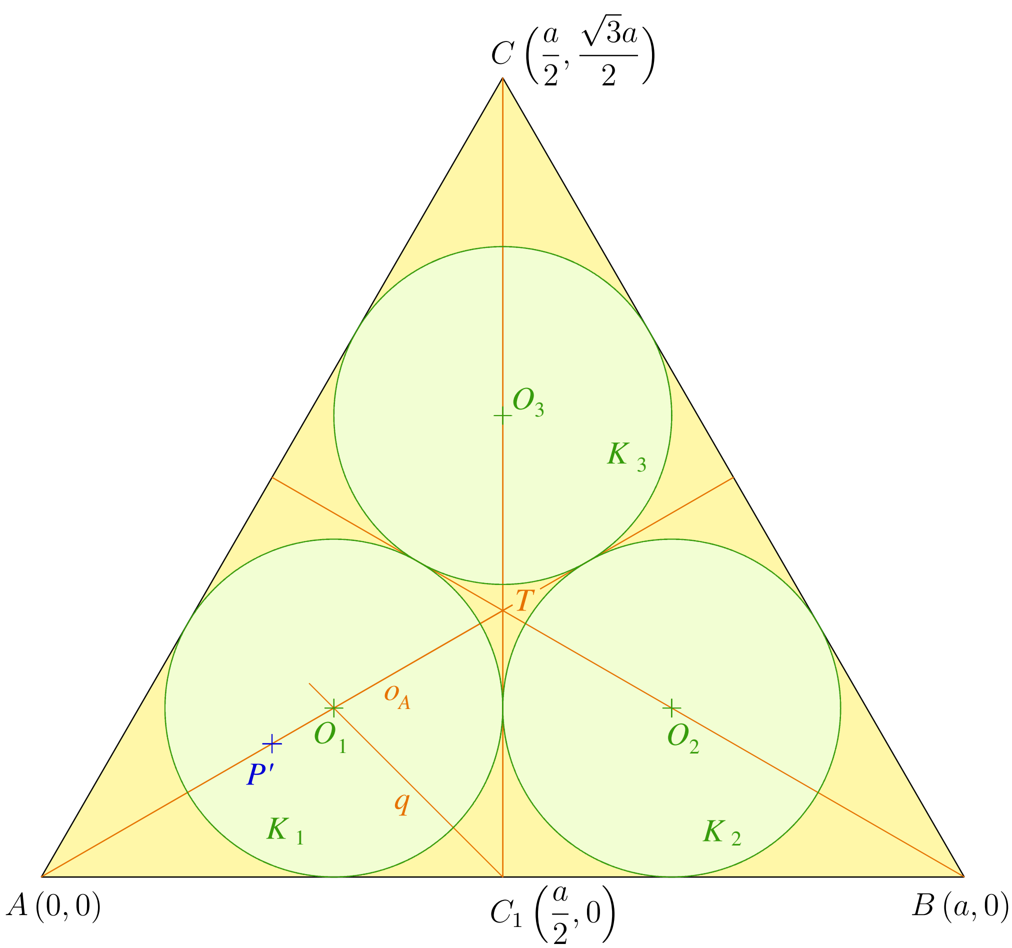


**Figure 5.**

Firstly, determine the areas of circles constructed by Malfatti (see Figure 5). Since we are considering an equilateral triangle, all the Malfatti circles are congruent.

If we denote the foot of the perpendicular from $C$ to the line $AB$ by $C_1$, then this point has coordinates $C_1 = (\frac{a}{2}, 0)$. Since the gravity center $T$ of $ABC$ divides the line segment $CC_1$ in ratio $2:1$, the centre of gravity $T$ has coordinates $T = (\frac{a}{2}, \frac{\sqrt{3}a}{6})$.

The Malfatti circle $K_1$ with center $O_1$ is the circle inscribed in the triangle $AC_1C$. Thus, $O_1$ lies on the axes of the angles $CAC_1 = CAB$ and $AC_1C$.

The direction vector $\vec{AT}$ of the axis $o_A$ of the interior angle $CAB$ of the triangle $ABC$ is $\vec{AT} = \left(\frac{a}{2}, \frac{\sqrt{3}a}{6}\right)$. Therefore, the parametric equations of $o_A$ are

$$\begin{aligned} o_A: \quad x &= \tfrac{a}{2}t, \\ y &= \tfrac{\sqrt{3}a}{6}t, \quad t \in \mathbb{R}\,. \end{aligned}$$

The direction vector $\vec{C_1O_1}$ of the axis $q$ of the angle $AC_1C$ is $(-1;1)$ and $q$ passes through the point $C_1 = (\frac{a}{2}, 0)$. Thus, the parametric equations of $q$ have the form

$$\begin{aligned} q: \quad x &= \tfrac{a}{2} - s\,, \\ y &= s, \quad s \in \mathbb{R}\,. \end{aligned}$$

We find the point $O_1$ as the intersection point of $o_A$ and $q$:

$$\begin{aligned} \tfrac{a}{2}t &= \tfrac{a}{2} - s\,, \\ \tfrac{\sqrt{3}a}{6}t &= s\,. \end{aligned}$$

We solve this system of linear equations using substitution:

$$\frac{a}{2}t + \frac{\sqrt{3}a}{6}t = \frac{a}{2},$$

and thus $3at + \sqrt{3}at = 3a$, giving

$$t = \frac{3a}{a\left(3+\sqrt{3}\right)}\,.$$

We deduce that

$$t = \frac{3}{3+\sqrt{3}} \cdot \frac{3-\sqrt{3}}{3-\sqrt{3}} = \frac{3-\sqrt{3}}{2}\,.$$

Hence, the coordinates of $O_1$ are $\left(\frac{(3-\sqrt{3})a}{2\cdot 2}, \frac{\sqrt{3}a\left(3-\sqrt{3}\right)}{6\cdot 2}\right) = \left(\frac{\left(3-\sqrt{3}\right)a}{4}, \frac{\left(\sqrt{3}-1\right)a}{4}\right)$.

The radius $r_1$ of the Malfatti circle $K_1$ is equal to the $y$-coordinate of $O_1$. Therefore, $r_1 = \frac{1}{4}\left(\sqrt{3}-1\right)a$ and the total area of three Malfatti circles is

$$S = 3\pi r_1^2 = 3\pi \left(\frac{\left(\sqrt{3}-1\right)a}{4}\right)^2 = \frac{3\pi\left(3-2\sqrt{3}+1\right)a^2}{16} = \frac{3\pi\left(2-\sqrt{3}\right)a^2}{8}\,.$$

Let us now pay attention to the second case (i.e., to the greedy arrangement) in which three circles of maximum area inside the triangle $ABC$ are gradually inscribed (see Figure 6).

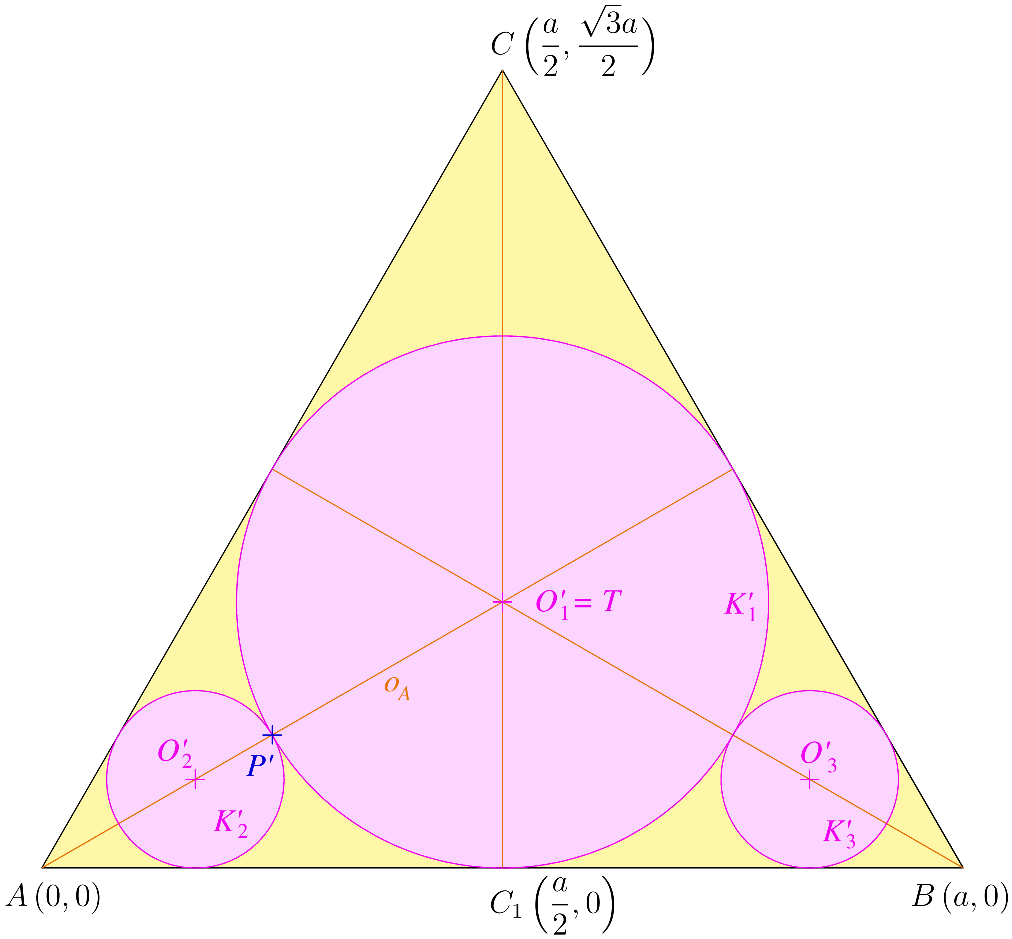


**Figure 6.**

The center $O'_1$ of the circle $K'_1$ with the maximum radius $r'_1$ ($K'_1$ is the incircle of $ABC$) coincides with the centre of gravity of the equilateral triangle $ABC$. Therefore, $O'_1 = \left(\frac{a}{2}, \frac{\sqrt{3}a}{6}\right)$ and it is obvious that $r'_1 = \frac{\sqrt{3}a}{6}$.

Thus, the area of $K'_1$ is

$$S'_{K_1} = \pi {r'}_1^2 = \pi \left(\frac{\sqrt{3}a}{6}\right)^2 = \frac{\pi a^2}{12}\,.$$

Let us determine coordinates of the centre $O'_2$ of the area-maximizing circle $K'_2$ which is tangent to the circle $K'_1$ and to two sides of triangle $ABC$ (see Figure 6).

The point of contact $P'$ where the circle $K'_1$ touches the circle $K'_2$ lies simultaneously on the axis $o_A$ of the angle $CAB$ and on the boundary $k'_1$ of the circle $K'_1$, i.e., on the circle with centre $O'_1 = \left(\frac{a}{2}, \frac{\sqrt{3}a}{6}\right)$ and radius $r'_1 = \frac{\sqrt{3}a}{6}$. Hence, the coordinates $(x_{P'}, y_{P'})$

of $P'$ satisfy simultaneously the equations of $o_A$

$$o_A: \begin{array}{rcl} x_{P'} &=& \frac{a}{2}t\,, \\ y_{P'} &=& \frac{\sqrt{3}a}{6}t, \quad t \in \mathbb{R}\,, \end{array}$$

and of circle $k_1'$

$$\left(x_{P'} - \frac{a}{2}\right)^2 + \left(y_{P'} - \frac{\sqrt{3}a}{6}\right)^2 = \left(\frac{\sqrt{3}a}{6}\right)^2.$$

After substituting, we obtain

$$\left(\frac{a}{2}t - \frac{a}{2}\right)^2 + \left(\frac{\sqrt{3}a}{6}t - \frac{\sqrt{3}a}{6}\right)^2 = \left(\frac{\sqrt{3}a}{6}\right)^2,$$

and, after rearranging, we have

$$\frac{a^2}{4}(t-1)^2 + \frac{a^2}{12}(t-1)^2 = \frac{a^2}{12}\,.$$

Hence,

$$4\,(t-1)^2 = 1\,.$$

We deduce that

$$|t-1| = \frac{1}{2}\,.$$

Hence, $t_1 = \frac{1}{2}$, $t_2 = \frac{3}{2}$. Since we need the intersection point of $o_A$ and $k_1'$ which has smaller $x$-coordinate of the two options, we will consider only the solution $t_1 = \frac{1}{2}$. So, $x_{P'} = \frac{a}{2} \cdot \frac{1}{2} = \frac{a}{4}$ and $y_{P'} = \frac{\sqrt{3}a}{6} \cdot \frac{1}{2} = \frac{\sqrt{3}a}{12}$. Therefore, $P' = \left(\frac{a}{4}, \frac{\sqrt{3}a}{12}\right)$.

The point $O_2'\left[x_{O_2'};\ y_{O_2'}\right]$ is the same distance from the point $P'$ and from the line $AB$ (thus, $O_2'$ lies on the parabola with focus $P'$ and directrix $AB$). Since this distance is its $y$-coordinate $y_{O_2'}$, we have

$$\sqrt{\left(x_{O_2'} - \frac{a}{4}\right)^2 + \left(y_{O_2'} - \frac{\sqrt{3}a}{12}\right)^2} = y_{O_2'}.$$

After squaring, we obtain

$$x_{O_2'}^2 - \frac{x_{O_2'} \cdot a}{2} + \frac{a^2}{16} + y_{O_2'}^2 - \frac{y_{O_2'} \cdot \sqrt{3}a}{6} + \frac{3a^2}{144} = y_{O_2'}^2,$$

and, consequently,

$$x_{O_2'}^2 - \frac{x_{O_2'} \cdot a}{2} + \frac{a^2}{16} - \frac{y_{O_2'} \cdot \sqrt{3}a}{6} + \frac{3a^2}{144} = 0.$$

The point $O_2'$ lies also on $o_A$, which implies

$$o_A: \begin{array}{rcl} x_{O_2'} &=& \frac{a}{2}t, \\ y_{O_2'} &=& \frac{\sqrt{3}}{6}at, \quad t \in \mathbb{R}. \end{array}$$

It follows that

$$\left(\frac{a}{2}t\right)^2 - \frac{\frac{a}{2}t \cdot a}{2} + \frac{a^2}{16} - \frac{\frac{\sqrt{3}}{6}at \cdot \sqrt{3}a}{6} + \frac{3a^2}{144} = 0\,;$$

that is,

$$\frac{a^2t^2}{4} - \frac{a^2t}{4} + \frac{a^2}{16} - \frac{a^2t}{12} + \frac{a^2}{48} = 0\,.$$

This gives

$$a^2\left(3t^2 - 4t + 1\right) = 0\,.$$

Since $a > 0$, we have

$$3t^2 - 4t + 1 = 0\,,$$

which has the following two solutions:

$$t_{1,2} = \frac{4 \pm 2}{6}\,; \quad \text{i.e.}\,, \quad t_1 = \frac{1}{3},\ t_2 = 1\,.$$

From these two options, we will choose the one for which the coordinate $x_{O_2'}$ is smaller ($O_2'$ lies on the mentioned parabola to the left of its focus $P'$). So, we consider only the solution $t_1 = \frac{1}{3}$, which gives us $x_{O_2'} = \frac{a}{2} \cdot \frac{1}{3} = \frac{a}{6}$, $y_{O_2'} = \frac{\sqrt{3}a}{6} \cdot \frac{1}{3} = \frac{\sqrt{3}a}{18}$. Hence, $O_2' = \left(\frac{a}{6}, \frac{\sqrt{3}a}{18}\right)$.

The radius $r_2'$ of $K_2'$ is equal to $y$-coordinate of $O_2'$, which implies $r_2' = \frac{\sqrt{3}a}{18}$. Thus, the area $S_{K_2'}$ of $K_2'$ is

$$S_{K_2'} = \pi r_2'^2 = \pi\left(\frac{\sqrt{3}a}{18}\right)^2 = \frac{\pi a^2}{108}\,.$$

The third inscribed circle $K_3'$ is congruent to $K_2'$ (see Figure 6). Thus, the sum $S'$ of areas of the circles $K_1'$, $K_2'$ and $K_3'$ is

$$S' = S_{K_1'} + 2 \cdot S_{K_2'} = \frac{\pi a^2}{12} + 2 \cdot \frac{\pi a^2}{108} = \frac{11\pi a^2}{108}.$$

Now, we can determine in which of the two cases the area of $ABC$ covered by the three circles is larger.

In the first case (the Malfatti circles), the ratio of the area $S$ of three circles to the area $S_\triangle$ of $ABC$ is

$$\frac{S}{S_\triangle} = \frac{\frac{3\pi\left(2-\sqrt{3}\right)a^2}{8}}{\frac{\sqrt{3}a^2}{4}} = \frac{3\pi\left(2-\sqrt{3}\right)}{8} \cdot \frac{4}{\sqrt{3}} = \frac{3\pi\left(2-\sqrt{3}\right)}{2\sqrt{3}} \approx 0.729\,.$$

In the second case (the greedy arrangement), the ratio is

$$\frac{S'}{S_\triangle} = \frac{\frac{11\pi a^2}{108}}{\frac{\sqrt{3}a^2}{4}} = \frac{11\pi}{108} \cdot \frac{4}{\sqrt{3}} = \frac{11\pi}{27\sqrt{3}} \approx 0.739\,.$$

In the first case, three circles cover approximately 72.9% of the area of the triangle. In the second case, it is approximately 73.9%, i.e. approximately one percent more. So even for an equilateral triangle, Malfatti's solution is not better.

The advantage of the greedy algorithm for the equilateral triangle was pointed out by H. Lob and H.W. Richmond in 1930 [3]. Moreover, Michael Goldberg showed in 1967 [2] that the greedy algorithm is always more optimal than the procedure presented by Malfatti. In 1994, Zalgaller and Los proved that the greedy arrangement is always the best solution of Malfatti's problem [6]. For more information about history of problem, see [1].

# References


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